Using mathematics to model the flight of a rocket

Introduction

For my Year In Industry, I am working at Chemring Countermeasures (CCM), a world leader in protecting air and naval platforms from the threats posed by heat-seeking and radar controlled missiles.

Mathematics is an essential tool in all areas of engineering, and here I consider the specific example of calculating a rocket's trajectory.

Derivation of Differential Equations

There are two equations which model the flight of the rocket; one which dictates its velocity and the other its rotation. This can be thought of as a "polar coordinate" method of modelling the rocket's trajectory.

1: Modelling velocity

From Newton's second law;

$$F = ma$$

$$a = \frac{dv}{dt} = \frac{F}{m}$$

Where **F** is the sum of all forces acting on the rocket;

$$F = thrust - drag - weight$$

$drag$ can be replaced by the drag equation, where C_d is the "coefficient of drag", ρ is air density, **A** the cross sectional area of the rocket and **v** its velocity.

$$drag = \frac{1}{2}C_d\rho Av^2$$

As weight acts vertically down, we must calculate the component of weight which acts parallel to the rocket's velocity;

$$weight = mgsin\theta$$

This gives our first equation (where **T** is equal to thrust). This function will be referred to as $f(v, \theta)$.

$$\frac{dv}{dt} = \frac{T - \left(\frac{1}{2}C_d\rho Av^2\right) - (mgsin\theta)}{m}$$

2: Modelling rotation

We use circular motion to model the rocket's rotation. The force acting perpendicular to the rocket's velocity is known as the centripetal force, and is equal to the component of weight acting in that direction.

$$\frac{mv^2}{r} = centripetal\ force = mgcos\theta$$
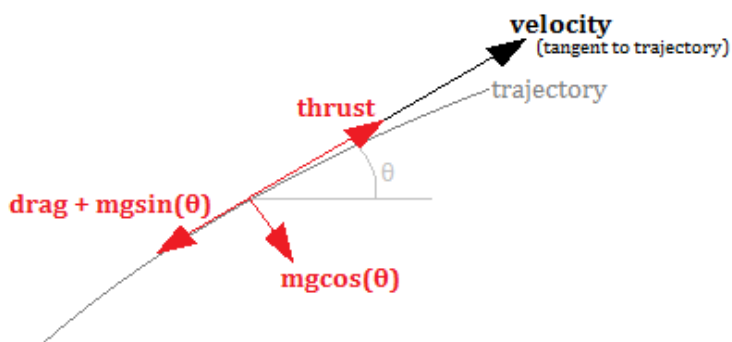
Since we know that;

$$angular\ velocity\ (\omega) = \frac{d\theta}{dt} = \frac{v}{r} \qquad (r = radius\ of\ curvature\ of\ trajectory)$$

This can be substituted into the previous equation to give;

$$mv\frac{d\theta}{dt} = mg cos\theta$$

Then simplified and rearranged for our second equation. Note that $\frac{d\theta}{dt}$ is negative as the centripetal force will reduce the angle of the rocket. This function will be referred to as $g(v,\theta)$.

$$\frac{d\theta}{dt} = \frac{-g cos\theta}{v}$$

Solution to Differential Equations

The Runge-Kutta method was used to obtain successive iterations for each differential equation.

For example, at time 1.74s, velocity was 121.43m/s and the rocket's angle 0.59862rad. The following table was used to calculate the angle and velocity at 1.76s, after 0.02s had passed (therefore $dt = 0.02$).

	Values
g	9.81 ms^-2
C_d	0.8
T	606.4 N
ρ	1.217 kgm^-3
m	30.3 kg
A	0.013273 m^2

v	Equal to	Numerical value	θ	Equal to	Numerical value
$\frac{dv}{dt}$	$f(v,\theta)$	11.34107	$\frac{d\theta}{dt}$	$g(v,\theta)$	-0.06674
k_{f1}	$dt \times f(v,\theta)$	0.226821	k_{g1}	$dt \times g(v,\theta)$	-0.00133
k_{f2}	$dt \times f(v + \frac{k_{f1}}{2}, \theta + \frac{k_{g1}}{2})$	0.226812	k_{g2}	$dt \times g(v + \frac{k_{f1}}{2}, \theta + \frac{k_{g1}}{2})$	-0.00133
k_{f3}	$dt \times f(v + \frac{k_{f2}}{2}, \theta + \frac{k_{g2}}{2})$	0.226812	k_{g3}	$dt \times g(v + \frac{k_{f2}}{2}, \theta + \frac{k_{g2}}{2})$	-0.00133
k_{f4}	$dt \times f(v + k_{f3}, \theta + k_{g3})$	0.226803	k_{g4}	$dt \times g(v + k_{f3}, \theta + k_{g3})$	-0.00133
$v_{1.76}$	$v_{1.74} + \frac{1}{6}(k_{f1} + 2k_{f2} + 2k_{f3} + k_{f4})$	**121.65m/s**	$\theta_{1.76}$	$\theta_{1.74} + \frac{1}{6}(k_{g1} + 2k_{g2} + 2k_{g3} + k_{g4})$	**0.59729rad**

Note; the spreadsheet used to calculate these figures works to many more significant figures than shown; slight rounding errors in the table may have occurred as a result.

Therefore, at time 1.76s, velocity was 121.65m/s and the rocket's angle 0.59729rad.

The Runge-Kutta method is very effective at calculating the trajectory of a rocket. However, it is an extremely lengthy process to obtain just one iteration; and so manually obtaining values for an entire flight would be impractical. In practice, a macro written in visual basic and incorporated into an excel spreadsheet is used to accurately predict the trajectory of a rocket.

The mathematics in this entry are adapted from my colleague Andy Barlow's excel trajectory calculator, originally from "Principles of Mechanics" (J. L. Synge / B. A. Griffith).