

MEI
STRUCTURED MATHEMATICS

MEI conference
University of Reading
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C3 COURSEWORK

*What is this coursework designed to do
and how do we prepare students for it?*

Presenter: Val Hanrahan

The coursework rationale is:

The aims of this coursework are that students should appreciate the principles of numerical methods and at the same time be provided with useful equation solving techniques.

The objectives are:

- that students should be able to solve equations efficiently, to any required level of accuracy, using numerical methods;
- that in doing so they will appreciate how to use appropriate technology, such as calculators and computers, as a mathematical tool and have an awareness of its limitations;
- that they show geometrical awareness of the processes involved.

The coursework requirements are:

Candidates will investigate the solution of equations using the following three methods:

- Systematic search for a change of sign using one of the three methods: decimal search, bisection or linear interpolation.
- Fixed point iteration using the Newton-Raphson method.
- Fixed point iteration after rearranging the equation $f(x) = 0$ into the form $x = g(x)$.

NB: A different equation must be used for each method. Students should avoid:

- Any equation used in Chapter 6 of the C3/C4 textbook.
- Trivial equations. For an equation to be non-trivial it must pass two tests:
 - It should be an equation they would expect to work on rather than just write down the solution (if it exists); for instance $\frac{1}{(x-a)} = 0$ is definitely not acceptable; nor is any quadratic (since there is the quadratic formula), or any polynomial expressed as a product of linear factors.
 - Constructing a table of values for integer values of x should not, in effect, solve the equation. Thus $x^3 - 6x^2 + 11a - 6 = 0$ (roots at $x = 1, 2$ and 3) is not acceptable.

Centres **may** provide students with a list of at least ten equations from which they can, if they wish, select those they are going to use, but any list must be forwarded to the moderator with the sample and changed for each examination session. Centres may refuse to issue a list on the grounds that candidates benefit from the mathematics they learn while finding their own equations.

The coursework assessment sheet is both detailed and prescriptive and should be issued to students as they start their coursework. Students should be encouraged to number their pages and enter in the 'comment' column the page number where they think each of the criteria is satisfied. This is helpful to them, as they can check that everything is included; to you, as you can see where they think it is, if it is not immediately apparent; and also to the moderator.

Domain	Mark	Description	Comment	Mark
Change of sign method (3)	1	The method is applied successfully to find one root of an equation.		
	1	Error bounds are stated and the method is illustrated graphically.		
	1	An example is given of an equation where one of the roots cannot be found by the chosen method. There is an illustrated example of why this is the case.		

Notes:

- Linear interpolation is not generally used.
- 3 decimal places is an acceptable level of accuracy.
- For both decimal search and interval bisection, the error bounds are contained within the method. The accuracy of any root should be stated in the form of error bounds ($x = 2.614 \pm 0.0005$) or solution bounds ($2.6135 \leq x \leq 2.6145$)
- The method must be illustrated for **their** equation, showing clearly where there is a change of sign. It is not sufficient to do a general illustration.
- Failure only usually occurs when the methods are applied blindly, for example manipulating the equation without first drawing a sketch graph, but it is possible to find examples where a graph drawn to a reasonable scale only appears to have one root in a unit interval, but in fact has more.

Domain	Mark	Description	Comment	Mark
Newton-Raphson method (5)	1	The method is applied successfully to find all the roots of a second equation.		
	1	All the roots of the equation are found.		
	1	The method is illustrated graphically for one root.		
	1	Error bounds are established for one of the roots.		
	1	An example is given of an equation where this method fails to find a particular root, despite a starting value close to it. There is an illustrated explanation why this has happened.		

Notes:

- This equation must be different from the first one.
- Roots should be found to at least 5 significant figures.
- The graphical illustration is for only **one root** and must relate to **their** equation.
- Error bounds must be established by looking for a change of sign and not just stated.
- Failure of the method requires that **a particular root** is not found, despite a starting value close to it. This means that either there is no root using that starting point, or the method converges to a different root from the one that was being sought.
- Again the illustration must relate to their equation.

Domain	Mark	Description	Comment	Mark
Rearranging $f(x) = 0$ in the form $x = g(x)$ (4)	1	A rearrangement is applied successfully to find a root of a third equation.		
	1	Convergence of this rearrangement to a root is demonstrated graphically and the magnitude of $g'(x)$ is discussed.		
	1	A rearrangement of the same equation is applied in a situation where the iteration fails to converge to the required root.		
	1	This failure is demonstrated graphically and the magnitude of $g'(x)$ is discussed.		

Notes:

- The magnitude of $g'(x)$ is **discussed**. It is not necessary to be able to differentiate $g(x)$, since comparing the gradient of the curve near the point where $x = g(x)$ with either a line of gradient 1 or a line of gradient -1, as appropriate, gives a very satisfactory discussion.
- For failure it is possible either to use the same rearrangement and a different root, or a different rearrangement and the same root, or a different rearrangement and a different root.

Domain	Mark	Description	Comment	Mark
Comparison of methods (3)	1	One of the equations used above is selected and the other two methods are applied successfully to find the same root.		
	1	There is a sensible comparison of the relative merits of the three methods in terms of speed of convergence.		
	1	There is a sensible comparison of the relative merits of the three methods in terms of ease of use with available hardware and software.		

Notes:

- Students should realise that their experiences are very limited, and should base any comparisons solely on these.
- They need to say what hardware and software they have been using, since any comments they make will be relative to this.

Domain	Mark	Description	Comment	Mark
Written communication (1)	1	Correct notation and terminology are used.		

Notes:

- A common error is to confuse 'function' and 'equation' by saying, for example 'I am going to use decimal search to solve the equation $f(x) = x^5 - 5x + 3$ '

Domain	Mark	Description	Mark
Oral communication (2)	2	Presentation Please tick at least one and give a brief report. Interview Discussion	

Notes:

- Each student must talk about the task. Topics for discussion may include strategies used to find suitable equations and explanations, with reference to graphical illustrations, of how the numerical methods work. An interview is also an ideal opportunity to pursue further any areas where there are errors or omissions in the coursework, although this will not affect the mark given in that domain.

Half marks may be awarded, but the overall total must be an integer so the assessor must make the decision to round up or down.

The total mark for this component is 18, so it constitutes 20% of the marks for this unit.

Points to be aware of when assessing C3 Coursework:

- Students may make up their own equations or choose from a list provided by their teacher. Any list provided should be sent with the scripts to the external moderator.
- Check that a different equation is used for each method.
- Students should show an appreciation of the difference between expressions, functions and equations. Repeated failure in this respect should incur some overall penalty, probably $\frac{1}{2}$ a mark.
- Equations for demonstrating failure should not be trivial, and a method has not failed if an initial table of values shows the roots. In general, equations with no roots should not be used as examples of failure.
- In ‘Comparison of methods’ it is reasonable to expect some quantitative statements such as NR takes 3 iterations, Change of Sign takes 4 etc. It is reasonable for students to base their judgements on the equations they happen to have used. However, good answers will refer, at least briefly, to general considerations such as potential difficulties in finding a rearrangement that works.

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*What I do to prepare my students for this
component in order to give a broad
learning experience.*

Presenter: Val Hanrahan

Preparatory work

We have tried starting the coursework in Year 12 after the examinations, but we have very little time (we have already nearly finished for the summer) and classes are always incomplete since traditionally this is the period allotted to educational visits and field trips.

For several years I have used a different approach which proved very successful. Although all our Year 12 students are lent graphics calculators for the duration of their A Level Mathematics course, not all had used them much for curve sketching – it depended very much on who was teaching them Pure Mathematics - so I devised a short ‘C3 Coursework Preparation’ exercise (Appendix 1). This exercise is given to them to complete over the summer break. They see it as useful, since they are being asked to find equations that they are going to use for their coursework, and I generally have a 100% return. A similar exercise could be performed using ‘Excel’, but I favour teaching them to use graphics calculators, since they are able to take them into their examinations, and this is an ideal opportunity to practise using them.

Why do I focus on polynomial equations?

At this stage students have been introduced to trigonometric and logarithmic curves, but have done very little with them. Using only polynomials, with which they are very familiar, allows them to concentrate on the numerical methods. They feel comfortable with polynomial graphs, having sketched them without calculators in C1, and are able to differentiate polynomial functions for those methods that require this. When this coursework was first introduced, many years ago, it was in the last of the three Pure Mathematics units, and in content it has changed very little since then. At that point students had met and differentiated many more functions, and often became more focussed on looking for ‘different’ equations to solve. This frequently led to less than satisfactory coursework, since they lost track of what they were supposed to be doing.

What happens next?

I encourage the use of different technology. It would be very easy to use ‘Autograph’ for the complete coursework, since it not only calculates values, but also sketches and illustrates the method for interval bisection, Newton-Raphson and ‘ $x = g(x)$ ’. However, it is my belief that to do this is not giving them as much breadth of experience as I would like.

Consequently, they do **decimal search** as their change of sign method, using the graph and table facilities from their calculator. Year 12 students who prefer to use a computer for the whole coursework will do this using ‘Excel’ - always an alternative for any of the coursework. (Appendix 2 for starting instructions)

The 'Autograph' package is ideal for the **Newton-Raphson** illustration, since the tangents drawn actually are tangents, not lines which would cut the curve at two or more points, but enable a lot of 'tangent sliding' to be shown on one sketch! It therefore emphasises just how quickly this method converges (Appendix 3 for starting instructions). Although this apparently does everything for them I require students to do the first iteration manually, which means that a polynomial equation must be used.

I also use 'Autograph' for $x = g(x)$, again because the illustrations are so good. When 'discussing' the magnitude of $g'(x)$ I tend to focus on what they can see from their graph, comparing the gradient of $y = g(x)$ with the gradient of $y = x$ (which is very straightforward), or with that of $y = -x + c$ for some suitable value of c that gives a line passing near to the relevant point of intersection (which is more difficult). Although the original function is a polynomial, $y = g(x)$ may be something that they do not yet have the skills to differentiate, since I do the coursework early in the C3 course.

The last two marks in the section '**Comparison of methods**' are the ones that cause most problems for both students and staff. Although they will have done some other textbook examples, and have their own fixed ideas from these, I encourage my students to focus on the first part of this domain, since they don't usually solve the same equation by a variety of methods and this is the only instance where they have some direct comparison. That being the case, their limited experience does not give them very much to write about, and a few clear sentences are sufficient to award the marks. Too frequently, however, students feel obliged to pad these out with a load of waffle! They do need to make clear to the assessor what hardware and software they have used.

The content of this handout is simply what **I** have done recently. I have taught the topic since its inception, first in the original Pure 3 (when there were only 3 Pure Mathematics units) and students' knowledge of different functions was much more advanced, then in Pure 2, when the Comprehension was introduced and became part of Pure 3, and currently in C3. My intention here is to pass on **my** thoughts on how this coursework can be done. It is by no means the only way, but may be of some help, particularly to those new to the topic.

The assessment sheet should be issued at the start, and students should be made aware of how prescriptive the mark allocation is. Emphasize that it is not at all like GCSE coursework (if they have done that) since, in my experience, few of my students enjoyed that and were not confident about exactly what was required to achieve a high mark.

Appendix 1 C3 Coursework Preparation

Your C3 coursework, which represents 20% of the marks available for that unit, is very different from the coursework you have done at GCSE. The assessment requires you to solve equations of your own choice by a variety of **numerical** methods, as opposed to the **analytical** (algebraic) methods that you have met so far.

Where analytical methods are possible, they are preferable because they give an **exact** solution. The numerical methods you will be using are all based on the ‘trial and improvement’ technique that you first used several years ago; using a numerical method you can only find a solution to a prescribed level of accuracy.

This worksheet is designed to help you identify suitable equations for your coursework by investigating the shape of different curves using your graphics calculator. Instructions refer to the Casio fx-9750, but similar operations apply to other models.

Background

The cubic and quartic curves that you sketched in C1 were ones that could be written in a factorised form: sketching $y = (x+1)(x-2)(x-3)$ was done by recognising the shape of a cubic with a positive x^3 term and identifying that this curve crossed the x -axis when $x = -1, 2$ and 3 , and the y -axis when $y = 6$.

Since this curve crossed the x -axis for rational values of x , the related equation, $(x+1)(x-2)(x-3) = 0$ had what we call an **analytical** solution.

What you need to find now are **equations that do not factorise**. One way of identifying cubic and higher order equations that do not have an analytical solution is to look for associated graphs that cross the x -axis at non-rational values. Your calculator graph cannot pick up non-rational values but, as a general rule, make sure that your graph doesn’t cross the x -axis at an integer value, and that the coefficient of your highest power of x is small.

Sketching $y = f(x)$

The instructions below are for the graph $y = x^5 - 5x + 3$, but can be adapted for any algebraic graph of the form $y = f(x)$.

- Choose the ‘GRAPH’ option from the main menu.
- Delete any equations in the list by highlighting them and pressing F2 (DEL), F1(YES). If there is only one equation you could type over it.
- Check that your calculator is set to graph functions of the form $Y =$. This will be on the top line of the display. If not, select the required type by pressing F3(TYPE), F1($Y =$)
- Enter the equation as $x^5 - 5x + 3$ (the $Y =$ is already there), using the x, θ, T button for x . Press EXE to store it. This automatically moves you to the next line. Press EXE again to draw the graph.

The scale that will be used is whatever you used for the previous graph, and may be totally unsuitable. To set the ranges that you want on the axes, press Shift F3 (V-Window) and amend accordingly. 'Scale' indicates the intervals at which the axes are marked. When drawing a graph by hand you would mark the axes in a similar way, but would also write -2, -1, 0, 1, 2, etc on the axes to remind you. These numbers are not on the calculator axes, so it is best to choose a scale that you will remember easily. Using too small a range initially may mean that you don't see enough of the curve and may miss some turning points or intersections with the x -axis.

A good starting range for most polynomials that you will want to graph is:

Xmin :	-5
max :	5
scale :	1
Ymin :	-50
max :	50
scale :	10

NB. If you are altering every entry, pressing EXE after you key in the new value is fine, but using that key if there is an entry that you don't want to change returns you to the equation. Using 'REPLAY' allows you to move down the values without changing them all. Press EXE at the end to return to the equation. Alternatively you can alter the scale using the 'Zoom' facility, but this only allows you to zoom in on something already on the screen. You can't access the rest of the graph if you haven't got a good enough range of values of x to start with.

F6 (G↔T) swaps between the graph and the equation displays.

Adding a second equation to the list will graph that equation on the same axes as the first. This is useful for simultaneous equations, but for this exercise you only want one equation on the screen at a time.

Exercise

For each of the graphs in the list, use the range suggested above to draw the graph on your calculator and then copy the graph onto file paper, (not graph paper) adding the scale to the axes. In each case, decide if those values of x and y give appropriate detail to the part of the curve where it crosses the x axis. If you then feel that this is not the case then:

- Say why you would want to change the x or y values or both
- Re-draw the graph on your calculator using different ranges until you find a really suitable one
- Copy that graph onto file paper, adding the scale to the axes.

1. $y = x^4 - 3x^3 - x^2 + 5$

2. $y = 2x^4 - x^3 + 7x - 12$

3. $y = 2 + 4x - 2x^2 - 3x^3$

4. $y = x^5 - x^4 + 2x^2 - 3x$

5. $y = 3x + x^2 - 2x^3 - 1$

Finding graphs for your coursework

Now make up **five** functions of your own. Some of these will lead to the equations you will use for your coursework. Your coursework uses three methods and ideally you should choose equations where the associated graph is going to cross the x -axis 2 or 3 times. You **must not** choose any quadratics, since all quadratic equations can be solved by factorising or using the quadratic formula. In addition, you must not choose a cubic that crosses the x -axis at an integer value, since using the factor theorem you could then find a linear factor, reducing the cubic to linear times quadratic and therefore solvable. Similarly, quartics crossing the x -axis at two integer values are excluded.

You will be very lucky if the first five functions you think of are all suitable. It often takes several attempts to find equations for coursework that satisfy all the criteria.

Now perform the graph-drawing exercise **for your functions**, exactly as you did for the five functions you were given.

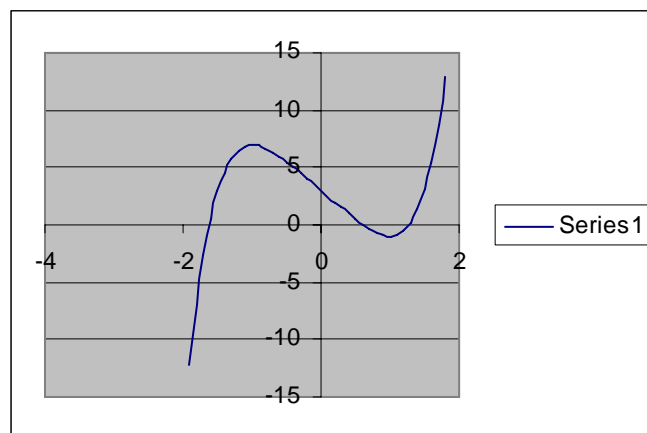
Appendix 2

Decimal Search using Excel to solve $x^5 - 5x + 3 = 0$

Start by choosing a minimum value of x ($x = -2$ here) and fill down using intervals of 0.1. The $f(x)$ values are calculated by entering $f(x)$ as $x^5 - 5x + 3$ and filling down.

There are three changes of sign in the range considered, with each of these giving an approximate location for a root of the equation $f(x) = 0$.

x	f(x)
-2	-19
-1.9	-12.261
-1.8	-6.89568
-1.7	-2.69857
-1.6	0.51424
-1.5	2.90625
-1.4	4.62176
-1.3	5.78707
-1.2	6.51168
-1.1	6.88949
-1	7
-0.9	6.90951
-0.8	6.67232
-0.7	6.33193
-0.6	5.92224
-0.5	5.46875
-0.4	4.98976
-0.3	4.49757
-0.2	3.99968
-0.1	3.49999
0	3
0.1	2.50001
0.2	2.00032
0.3	1.50243
0.4	1.01024
0.5	0.53125
0.6	0.07776
0.7	-0.33193
0.8	-0.67232
0.9	-0.90951
1	-1
1.1	-0.88949
1.2	-0.51168
1.3	0.21293
1.4	1.37824
1.5	3.09375
1.6	5.48576
1.7	8.69857
1.8	12.89568
1.9	18.26099
2	25



To graph the expression $y = x^5 - 5x + 3$, highlight the cells that you want to use (in this case x from -1.9 to 1.8 and the corresponding values of y) and select the chart wizard (next to A-Z and looks like a histogram). The option you want is XY scatter, and then choose the sub-type with the curve drawn without showing individual points. Continue by pressing 'Next' until you are ready to 'Finish'

This gives you the shape of the curve, but is not as good as 'Autograph' for the detail you need to include with the coursework.

From the graph above, you can see that there are three roots in the interval considered, together with their approximate positions. Also, from the shape of the curve it looks as if there will only be these three roots.

For the root in (-1.7,-1.6), start with $x = -1.7$ and fill down using intervals of 0.01 until a sign change is obtained etc.

Appendix 3

Autograph and C3 coursework

The Basics:

Autograph is an advanced graph drawing package that will make the graphical side of the C3 coursework much simpler than drawing by hand or using Excel. It is currently on the school network, but if working from home you can download a 30-day free trial. It is probably best to leave this until the last 30 days before your coursework deadline.

Start by familiarising yourself with the important menu options:

Adding an equation (the button with an = and + sign on it) Pressing this button lets you enter equations to plot, using the ^ key to register a power. Alternatively use 'Equation' from the tool bar.

Changing the scale (the spanner button) Pressing this button lets you adjust the minimum and maximum x and y values. Alternatively use 'Axes' from the tool bar.

Zooming Clicking on the magnifying glass button (+ or -) lets you zoom in or out repeatedly. The same result may also be achieved by pressing the + and - keys at the far right of the keyboard. The buttons to the right of the magnifying glass are perhaps more useful as they enable zooming in to an area you've first drawn a box around. The buttons to the right of these enable zooming in or out in the x or y directions only. Alternatively you can select the area you want by just editing the axes.

Undoing a zoom There is an undo button (between the hand and the arrow buttons), but it will only undo your most recent zoom. You may also return to the default axes you started off with (the button just to the left of the degrees button).

Selecting a curve Positioning the mouse alongside a curve until a black arrow appears and then left-clicking the mouse turns the curve black and selects it for analysis. To select a second curve, you must first hold down the SHIFT key on the keyboard, and then repeat the above.

Right-clicking the mouse This operation gives several additional menu options, especially once one or more curves have been selected.

Slow motion drawing Press the button that looks like a tortoise. All curves are now drawn in slow motion. The buttons alongside this one allow for pausing etc.

Copying into WORD Clicking on PAGE and the COPY PAGE enables you to paste the graph, its annotation and the relevant equations into WORD.

Display of results The results box is situated to the left of the spanner button. Pressing it will give a display of all results found during the session.

There are many other facilities, but if you are familiar with all the above, you will be in a position to add graphs to your coursework.