M.E.I. Conference 2010

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Topics in Undergraduate Mathematics
Continuity and Differentiability

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You have to start with aims

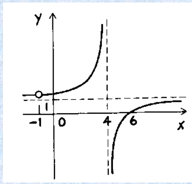
- To review these ideas in an informal manner
- To discuss why these ideas are important and useful
- To discuss how these ideas are met at school
- To use GeoGebra a bit
- To recommend some books!
- Please don't expect anything coherent: this is more a grab-bag of ideas
- Also I have forgotten quite a lot of my "undergraduate mathematics" – it's been a long time
- I hope the hour will expand to fit the material

Let's do some maths

- AS Further Pure Mathematics by C.Berry, T.Heard and D.Martin (MEI FP1)
- Chapter 3: "Graphs and inequalities"
- Page 86, Ex.3A, Q12
- Sketch the graph of $y = \frac{x^2 - 5x - 6}{(x+1)(x-4)}$
- Have a go!
- You don't need graph paper…

Sir, can we look in the back?

- Yes…here's what it has as the answer



- Undefined at $x = 4$. What about at $x = -1$?
- Is it discontinuous at $x = -1$?

Definition of continuity…

- and why it's important.
- Oxford University 1983

- "Analysis" in the first term of the first year

Definition of continuity

- Tom Apostol, "Mathematical Analysis":
- A function $f(x)$ is continuous at $x = c$ if $\forall \varepsilon > 0$ $\exists\, \delta > 0$ s.t. $|x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon$
- I think this definition is due to Weierstrass although a form of the definition was first given by Bolzano in 1817
- Need for a reliable definition of continuity arose from a desire to formulate and prove the theorems of calculus in a rigorous manner

Just to confuse the issues…

- We also had in the first year:
- the **limit definition**: $f(x)$ is continuous at $x = c$ if $\forall$ sequences $x_n \to c$, $f(x_n) \to f(c)$
- the **topological definition**: if $f : X \to Y$ where X and Y are topological spaces, f is continuous if and only if $\forall$ open sets $V \subseteq Y$, $f^{-1}(V) = \{x \in X \mid f(x) \in V\}$ is open.
- Are you with me?

No, we didn't get it either.

Brief survey of where it comes now

- it = continuity of functions
- Oxford: First year, Hilary Term (Analysis II)
- Warwick: First year, first and second terms (MA131 Analysis)
- Durham: First year, all year (Core B1)
- Birmingham: Second year

It is important

- The ideas of continuity and **limit** underpin the differential calculus
- Basic ideas: Newton and Leibnitz of course!
- They used **infinitesimals**
- Archimedes had discussed infinitesimals in his Method of Mathematical Theorems:
- x is infinite if $|x| > 1$, $|x| > 1 + 1$, $|x| > 1 + 1 + 1$…
- and infinitesimal if $x \neq 0$ and a similar set of conditions holds for $1/x$ and reciprocals of positive integers

Infinitesimals under attack!

- Bishop George Berkeley (1685-1753), Bishop of Cloyne, Ireland
- In 1734 Bishop Berkeley published *The Analyst, or A Discourse Addressed to an Infidel Mathematician*

Infinitesimals under attack!

XVI. If you assume at first a Quantity increased by nothing, and in the Expression $x + o$, o stands for nothing, upon this Supposition as there is no Increment of the Root, so there will be no Increment of the Power; and consequently there will be none except the first, of all those Members of the Series constituting the Power of the Binomial; you will therefore never come at your Expression of a Fluxion legitimately by such Method. Hence you are driven into the fallacious way of proceeding to a certain Point on the Supposition of an Increment, and then at once shifting your Supposition to that of no Increment. There may seem great Skill in doing this at a certain Point or Period. Since if this second Supposition had been made before the common Division by o, all had vanished at once, and you must have got nothing by your Supposition. Whereas by this Artifice of first dividing, and then changing your Supposition, you retain 1 and nx^{n-1}. But, notwithstanding all this address to cover it, the fallacy is still the same. For whether it be done sooner or later, when once the second Supposition or Assumption is made, in the same instant the former Assumption and all that you got by it is destroyed, and goes out together. And this is universally true, be the Subject what it will, throughout all the Branches of humane Knowledge; in any other of which, I believe, Men would hardly admit such a reasoning as this, which in Mathematics is accepted for Demonstration.

Infinitesimals survive

- Infinitesimals continued to be used to produce correct results
- Calculus was given a formal mathematical foundation by Weierstrass and others using the idea of a **limit**
- In the 20th century it was found that infintesimals could be treated rigorously, e.g. the work of Abraham Robinson (1918-1974) in **non-standard analysis**
- But let's not go there.

Mathematical Interlude

- How would you find the derivative of $f(x) = 2^x$ at $x = 0$ from first principles? Discuss…
- Let's use Excel to investigate!
- Oh dear!
- Try $h = 10^{-12}$ on a Casio fx-83ES
- Isn't this "small"?
- Aside: may I commend to you the MEI Numerical Methods unit
- In MEI FP1, $x = 100$ is officially "large"

What is a limit?

- The task of finding $\lim_{x\to 0} f(x)$ is the same as finding a number L such that if you place a dot at $(0, L)$, the parts of the graph of $y = f(x)$ for $x > 0$ and $x < 0$ are joined up, so the resulting graph is smooth or **continuous** across the vertical line through $(0, L)$.
- Compare this with those εs and δs…is this "better"?
- There is a problem…

How to annoy small children

- There are NO numbers "either side" of zero!
- Ask Year 7 "What is the smallest number larger than zero?"
- **Principle of Trichotomy**: given two numbers a and b, then $a = b$ or $a < b$ or $a > b$. (True?)
- Suppose $x \neq 0$ is "next to" 0. Then $x < 0$ or > 0.
- Either way, $x/2$ is closer to 0 than x.
- Oh dear…

Professor E. McSquared

- If I had had this book while at university, I would have understood what all those epsilons and deltas were doing much more easily
- It would be a good acquisition for any school library
- It is available direct from the publishers
- I am not on commission from them

What about the function we started with?

- Another formulation is
- $f(x) = \frac{x-6}{x-4}$ if $x \neq -1$
- and undefined if $x = -1$.
- From the graph, it certainly looks like $f(x)$ wants to be 1.4 if $x = -1$.
- So let $\overline{f}(x)$ be $f(x)$ if $x \neq -1$, and 1.4 if $x = -1$.
- Now we prove that $\overline{f}(x)$ is continuous at -1.

Typing into PowerPoint

- is not easy.
- So I have a **handout**.
- Please tell me if it's wrong. This is the first epsilon/delta proof I've done for probably 22 years…

School Mathematics

- How relevant is all this to school maths?
- When do pupils first meet:
  - infinity?
  - discontinuous graphs?
  - limits and convergence?
- What do pupils think undergraduate maths is all about?

Questions about infinity

- Quote from Year 7: "Sir, can we do infinity today?"
- G.H.Hardy (A Course of Pure Mathematics, 1908):
  - The class of British subjects is a finite class
  - The class of positive integers is not finite but infinite
- Are there more integers than positive integers?
- Yes…so $\infty = 2\infty + 1 \Rightarrow \infty = -1$?
- Are there more "decimals between 0 and 1" than integers?
- "Good and bad" kinds of infinity

Countable and uncountable sets

- Graeme Segal, Oxford 1983

Discontinuous graphs

- Year 8? $y = 1/x$
- Year 10? $y = 1/(x - 3)$ (translations)
- Year 10/11? $y = \tan x$
- FP1 text p.93: $y = \dfrac{x+2}{(x-2)(x+1)}$
- Approaching *x*-axis from "above" or "below"
- FP1 text p.86: $y = \dfrac{x-3}{(x-4)^2}$
- "Double" asymptote

Discontinuous graphs

- Decision 1 W08: $f(x) = [x]$

The following algorithm (J. M. Oudin, 1940) claims to compute the date of Easter Sunday in the Gregorian calendar system.
The algorithm uses the year, y, to give the month, m, and day, d, of Easter Sunday.
All variables are integers and **all remainders from division are dropped**. For example, 7 divided by 3 is 2 remainder 1. The remainder is dropped, giving the answer 2.

- Core 3 and Numerical Methods: do we consider whether a graph is discontinuous when we use the change of sign method to locate a root?

Limits

- $y = 1/x$ as $x \to \infty$: when considered?
- An intuitive idea is easy, but what do we really mean by "$x \to \infty$"?
- $f(x) \to l$ as $x \to \infty$ if $\forall\, k, \exists\, x_0$ s.t. $|f(x) - l| < k$ $\forall\, x > x_0$
- "Formal" limits: differentiation from first principles (Add Maths Year 11/Year 12?)
- "Limiting Processes": option in MEI P6. In other A level Further Maths specifications?

AQA FP3

- W09 Q6

6 The function f is defined by $f(x) = e^{2x}(1+3x)^{-\frac{2}{3}}$

(c) Find

$$\lim_{x \to 0} \frac{1 - f(x)}{x \ln(1 + 2\sin x)}$$ *(3 marks)*

S.M.P. Further Maths (-1996)

- S.M.P. Extensions of Calculus, C.Goldsmith & D.Nelson

(1) Consider the graph of $y = \dfrac{3x^2 + 2x}{x - 9}$

- Have a go?

Oblique asymptotes

- Argument in the text goes like this
- $y = 3x + 29 + \dfrac{261}{x - 9}$
- $y = 3x + 29 + \dfrac{\frac{261}{x}}{1 - \frac{9}{x}}$
- As $x \to \infty$, the top $\to 0$ and the bottom $\to 1$
- It follows that $y = 3x + 29$ is an asymptote
- So can we say that $y \to 3x + 29$ as $x \to \infty$?

(2) Improper integrals

- $\int_{-2}^{3} x^{-1}\,dx = \left[\ln|x|\right]_{-2}^{3} = \ln 3 - \ln 2 = 0.405...?$
- No! Core 3: ∞ - ∞ = 0.405…
- But $\int_{-1}^{8} x^{-\frac{2}{3}}\,dx$ exists. Why?
- $\int_{-1}^{u} x^{-\frac{2}{3}}\,dx = \left[3x^{\frac{1}{3}}\right]_{-1}^{u} = 3u^{\frac{1}{3}} + 3 \to 3$ as $u \to 0$ from below
- $\int_{u}^{8} x^{-\frac{2}{3}}\,dx = \left[3x^{\frac{1}{3}}\right]_{u}^{8} = 6 - 3u^{\frac{1}{3}} \to 6$ as $u \to 0$ from above
- so the value of the integral is 9

Do we still do these things?

- MEI Statistics 3, W08, Q1(a)

The time (in milliseconds) taken by my computer to perform a particular task is modelled by the random variable T. The probability that it takes more than t milliseconds to perform this task is given by the expression $P(T > t) = \frac{k}{t^2}$ for $t \geqslant 1$, where k is a constant.

(i) Write down the cumulative distribution function of T and hence show that $k = 1$. [3]

(ii) Find the probability density function of T. [2]

(iii) Find the mean time for the task. [3]

$$\mu = \int_1^{\infty} t\,f(t)\,dt = \int_1^{\infty} \frac{2}{t^2}\,dt$$

$$= \left[\frac{-2}{t}\right]_1^{\infty}$$

$$= 0 - (-2) = 2$$

Convergence of sequences/series

- Year 7: intuitive ideas: Zeno's paradox, etc. Recurring decimals: 0.9 recurring = 1? (What's the difference? "Infinity of zeros and then 1")
- GCSE: "formal" proof that e.g. 4/11 = 0.36 recurring
- Year 12: Core 2, Sequences and Series: Freddie the Frog
- Year 13: Core 4: Binomial theorem for *n* not a positive integer. FP2: Maclaurin series. S2: motivating the Poisson distribution by using the expansion of e^x
- Oxford Interview 1982: bouncy ball, coefficient of restitution 0.6…does it stop?

Some software

- GeoGebra is amazing but I don't know how to use it properly yet
- I'm going to try to use it to illustrate some "Hennings Functions"
- Actually Chris Sangwin did it for me
- There is a whole strand on GeoGebra at this conference
- You can probably do a lot of this on Autograph™

Hennings functions

- When we were at university, we were taught by a brilliant young Junior Research Fellow, Mark Hennings
- He went on to be admissions tutor at Sidney Sussex, Cambridge, but was last heard of teaching at Rugby School
- He solves the hard problems in the Mathematical Gazette
- I can't find a picture of him
- Tutorials could be frightening. He used to destroy our "proofs" by producing "Hennings functions"

Hennings function #1

- This illustrates the function

$$f(x) = \left(1 + \frac{1}{n^x}\right)^{n^x}$$

- Chris Sangwin did it on GeoGebra for me

x / n	1	2	5	10	20	30
0	2	2	2	2	2	2
5	2	2.67699	2.717847	2.718268	2.718281	2.718282
10	2	2.716956	2.718282	2.718282	2.719489	2.854569
20	2	2.718281	2.705438	1	1	1
30	2	2.718282	1	1	1	1

Hennings function #2

- This illustrates the function

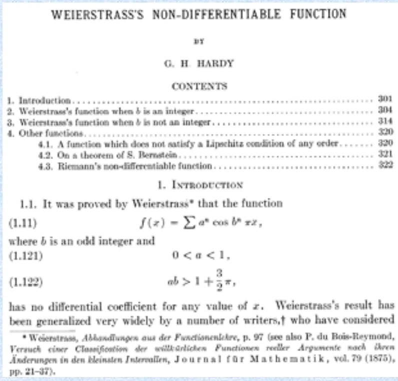
$$f_n(x) = \sum_{r=1}^{n} \frac{\sin(r^2 x)}{r^2}$$

- and the spreadsheet capability of GeoGebra
- Again, Chris showed me how to do it
- It's very crinkly. Zoom in. I think as *n* gets bigger it tends to a function which is continuous everywhere, but differentiable nowhere

A sum of smooth things can be crinkly

- Weierstrass thought of one as well. The paper on the next page looks like it's advertising a Victorian circus act.

Weierstrass' non-differentiable function



WEIERSTRASS'S NON-DIFFERENTIABLE FUNCTION

BY

G. H. HARDY

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1. INTRODUCTION

1.1. It was proved by Weierstrass* that the function

(1.11) $$f(x) = \sum a^n \cos b^n \pi x,$$

where b is an odd integer and

(1.121) $$0 < a < 1,$$

(1.122) $$ab > 1 + \frac{3}{2}\pi,$$

has no differential coefficient for any value of x. Weierstrass's result has been generalized very widely by a number of writers,† who have considered

* Weierstrass, *Abhandlungen aus der Functionenlehre*, p. 97 (see also P. du Bois-Reymond, *Versuch einer Classification der willkürlichen Functionen reeller Argumente nach ihren Änderungen in den kleinsten Intervallen*, Journal für Mathematik, vol. 79 (1875), pp. 21–37).

Further reading

- Wikipedia for some of the history!
- http://www.maths.tcd.ie/pub/HistMath/People/Berkeley/Analyst/Analyst.pdf
- "Understanding Infinity" by Tony Gardiner
- "Using Counter-examples in Calculus" by John Mason and Sergiy Klymchuk
- "A Course of Pure Mathematics" by G.H.Hardy
- Various school textbooks (sometimes the older the better, e.g. SMP Further Maths and things by Durell and Porter)
- and, of course, Professor E. McSquared's Calculus Primer (Expanded Intergalactic Version)
- http://www.math.sjsu.edu/~swann/mcsqrd.html
- and "The Theory of Differential Calculus" by Howard Swann (internet download)

48 slides in 60 minutes?

- Thank you for watching!
- I thoroughly enjoyed preparing this: it brought back many memories, most of them good
- I would like to thank Dr. Chris Sangwin of the University of Birmingham for many ideas
- amrogers1@gmail.com

Topics in Undergraduate Mathematics: Continuity and Differentiability

An attempt at a proof that $f(x) = \frac{x-6}{x-4}$ $(x \neq -1)$, 1.4 $(x = -1)$ is continuous at $x = -1$.

What we want is to find a $\delta(\varepsilon)$ such that

$$|x+1| < \delta(\varepsilon) \Rightarrow 1.4 - \varepsilon < \frac{x-6}{x-4} < 1.4 + \varepsilon$$

$$\Rightarrow \quad -\varepsilon < \frac{x-6}{x-4} - 1.4 < \varepsilon$$

$$\Rightarrow \quad -\varepsilon < \frac{x-6-1.4(x-4)}{x-4} < \varepsilon$$

$$\Rightarrow \quad -\varepsilon < \frac{-0.4(x+1)}{x-4} < \varepsilon$$

$$\Rightarrow \quad \left|\frac{0.4(x+1)}{x-4}\right| < \varepsilon$$

Suppose $\delta(\varepsilon) < ½$. Then $|x+1| < ½$ and

$$-½ < x + 1 < ½$$

$$\Rightarrow \quad -1.5 < x < -0.5$$

$$\Rightarrow \quad -5.5 < x - 4 < -4.5$$

$$\Rightarrow \quad 4.5 < |x-4|$$

$$\Rightarrow \quad \left|\frac{0.4(x+1)}{x-4}\right| < \frac{0.4|x+1|}{4.5} = \frac{4|x+1|}{45}.$$

So if we take $\delta(\varepsilon) = \min\{\frac{1}{2}, \frac{45}{4}\varepsilon\}$, we can guarantee that

$$|x+1| < \delta(\varepsilon) \Rightarrow \left|\frac{0.4(x+1)}{x-4}\right| < \varepsilon$$

$$\Rightarrow \quad 1.4 - \varepsilon < \frac{x-6}{x-4} < 1.4 + \varepsilon,$$ which is what we wanted.

Phew!