Approaches to teaching algebraic concepts at A level

MEI Conference 2010

Doug French

- This session draws very heavily from Doug French's book 'Teaching and Learning Algebra'.
- It will cover
  - Why study algebra?
  - Early misconceptions with algebra
  - Good algebraic exercises
  - Algebraic Fractions
  - **When is too much algebra?**
  - Curve Sketching
  - Trigonometry
  - Vectors

What are the problem with algebra?

- Students aren't motivated to study it
- Sixth form and HE concerns about lack of fluency in algebraic skills (going back to tackling the Mathematics Problem LMS/IMA/RSS 1995)
- We've seen moves to make algebra more accessible and attractive (ICT)
- Then swings back to repetitive exercises
- But more is needed, particularly as problem solving skills are weak too.

Why study algebra?

Two reasons can be advocated

- It's useful
- It's interesting

However

- Only directly useful to a minority of students
- Indirectly useful to many to develop thinking skills but other subjects do that too.
- We need to focus on – **because it's interesting**

How should students learn algebra?

- Learn skills then attempt to apply them to problems?
- Start with the problems and then acquire the skills necessary to solve them?

Good teaching will use a blend of these. The ratio of the two will have a strong influence on students' perceptions and performance.

But there's more to it than this.

What is algebra?

An economical and consistent symbol system to represent expressions and relationships which are then used in formulating arguments concerned with

- Prediction (evaluating a function)
- Problem solving (solving an equation)
- Explanation and proof (algebraic identities)

Early misconceptions

- Juxtaposition

  27 2p 23 2g 2½ 2A

- Simplification

Student often acquire the erroneous idea that letters stand for objects.

So then 2k + k + 4 = 7 and what is bc?

Handling student errors

- In the case of an error like 2k + k + 4 = 7 it's not sufficient to tell a student that they are wrong and then make reference to 'collecting like terms'
- The student need to know why they are wrong.

Algebra and Number

- If k is understood to be a number then substituting values makes it clear when two expressions are not equivalent.
- Be aware of the **numerical backdrop to algebraic ideas!**
- **Key Point 1: Many problems in algebra come from distancing algebra from its numerical basis.**

What is algebra?

An economical and consistent symbol system to represent expressions and relationships which are then used in formulating arguments concerned with

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- Explanation and proof (algebraic identities)

Letters – specific and general

- A letter as a specific unknown is the interpretation with equation solving
- But because students learn this first they get stuck with it and find difficulty with the letter as a generalised number.

Back to equation solving

- I am thinking of a number….
- Short step to 2x – 5 = 7
- Helping students to make this translation in each direction is often neglected because greater emphasis is given to the process of solution.

Formula and Functions Approach

- Emphasising equations in the early stages encourages the view that letters always stand for a specific unknown number.
- The formulae approach is better because it emphasises the idea of a function and algebra as a way to express relationships between numbers.

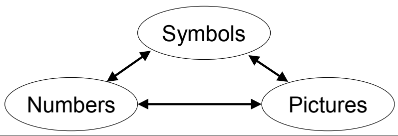
Simple Example

- P = 2l + 2w = l + w + l + w = 2(l + w)

Length, l	Width, w	Perimeter P
2	3	10
3	3	12
3	6	18

Patterns

- The essence of algebra is clearly the power that symbolic representation gives.
- In order to develop the facility to capitalise on this power the links between symbolic forms, number forms and picture forms need to be established and constantly reinforced



Connect Algebra to Number: 21 x 21

Example concerned with expanding brackets:

- How might you calculate 21 x 21?

Maybe you would calculate 20 x 20.

Then you realise the for 21 x 20 you need and extra 20.

Then you realise that for 21 x 21 you need an extra 21

So the final answer is 400 + 20 + 21.

Connecting Algebra to Number

- This leads directly to

$(n + 1)(n + 1) = n^2 + n + (n + 1)$
$= n^2 + 2n + 1$

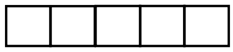
Algebra to describe relationships

- Ask students to tell you the third odd number
- Ask students to tell you the 10th odd number
- Ask students to tell you the 20th or 100th odd number.
- So 2n – 1 is the nth odd number.
- Then what about ‘which odd number is 283?’.

Adding two consecutive odd numbers

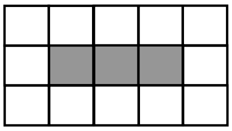
- The sum of two consecutive odd numbers is always a multiple of 4.
- Prove it.

Similar exercises



- How many matchsticks for 5 squares?
- How many for 10 squares?
- How many for n squares?
- How many squares can you make like this with 100 matches?

Similar exercises



- How many white tiles for 3 shaded tiles?
- How many for 10 shaded tiles?
- How many for n shaded tiles?
- If there are 50 white tiles how many shaded tiles are there?

The magic number problem

- Choose four randomly chosen, distinct, single digit numbers.
- Successively add each pair to eventually generate a single 'magic number'.

3 8 5 2

11 13 7

24 20

44 ← Magic Number

The magic number problem

Discuss the effect of changing the initial four numbers

- When will the magic number stay the same?
- How many magic numbers are possible with a given set of four numbers?
- How can you predict what the magic number will be?

$a \quad b \quad c \quad d$

$a+b \quad b+c \quad c+d$

$a+2b+c \quad b+2c+d$

$a+3b+3c+d$

Curve Sketching

- In a traditional approach a lot of emphasis is put on sketching a line/curve from its equation.
- The techniques involved for this often skip over the idea that an equation involving y and x expresses the relationship between the x and y coordinates of points on that curve!

For example

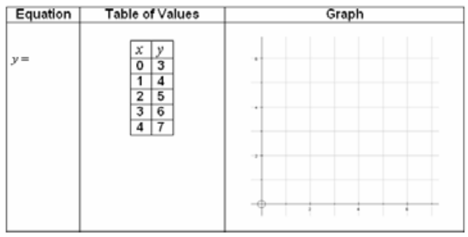
To sketch $y = 2x + 1$

Gradient is 2 and intercept with the y-axis is 1 therefore…

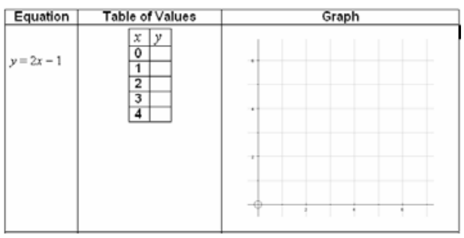
Functions and graphs

- Students need to understand the links between the equation, the table of values or set of coordinates and to be able to move fluently between these representations.
- In an advanced module such as Numerical Methods these ideas are so important. Without them the learning experience for the student is so much less rich.

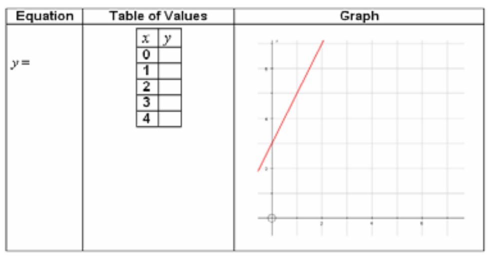
Equation, table, graph



Equation, table, graph



Equations, table, graph



For example

To sketch $y = \frac{3x+1}{x-1}$

When x = 0….
If y = 0 then etc.
There is a vertical asymptote, x = 1 etc.
If x > 1 then 3x + 1 > 0 and x – 1 > 0 ….

$$y = \frac{3(x-1)+4}{x-1} = 3 + \frac{4}{x-1}$$

What's the problem?

- Whenever we are doing this we should be asking specific questions that require numerical evaluation to emphasis the fundamental point that the equations gives the relationship between the x and y coordinates.
- This can actually be a very useful approach in more complex curve sketching

Example

$$y = \frac{3x+1}{x-1}$$

Example (FP1)

$$y = \frac{2x^2+3x+1}{x^2-3x+2}$$

Then algebra..

- Algebraic tricks to exam behaviour around asymptotes can follow after the numerical experimentation.
- They will be understood and appreciated so much more for it.

Algebraic Fractions

- Equivalence is one of the most fundamental ideas with fractions; different forms like

$$\frac{3}{4}, \frac{9}{12}, \frac{15}{20}$$

all have the same value.
- Understanding this and proper and improper fractions is essential in order to add and subtract fractions and to simplify results:

$$\frac{5}{6} + \frac{5}{12} = \frac{10}{12} + \frac{5}{12} = \frac{15}{12} = \frac{5}{4} = 1\frac{1}{4}$$

Common errors

$$\frac{x+2}{x+4}=\frac{1}{2} \qquad \frac{x+2}{x+4}=\frac{x+1}{x+2}$$

- Substituting simple numbers into such fractions can highlight the error
- Focus discussion on how, for simplification of this kind, genuine common factors must be sought.

Adding and subracting algebraic fractions

$$\frac{1}{2}+\frac{2}{1}=2\frac{1}{2},\ \frac{2}{3}+\frac{3}{2}=2\frac{1}{6},\ \frac{3}{4}+\frac{4}{3}=2\frac{1}{12},\ \frac{4}{5}+\frac{5}{4}=2\frac{1}{20}$$

$$\frac{n}{n+1}+\frac{n+1}{n}=2+\frac{1}{n(n+1)}$$

- This now provides a meaningful context for learning how to manipulate algebraic fractions
- Compare the algebraic manipulation with the numerical manipulation

Common Denominators

$$\frac{5}{6}+\frac{4}{15} = \frac{75+24}{90} = \frac{99}{90} = \frac{11}{10}$$

$$\frac{5}{6}+\frac{4}{15} = \frac{25+8}{30} = \frac{33}{30} = \frac{11}{10}$$

Common Denominators

$$\frac{1}{(n+1)(n+2)}+\frac{1}{(n+2)(n+3)}$$
$$=\frac{(n+2)(n+3)+(n+1)(n+2)}{(n+1)(n+2)^2(n+3)}$$
$$=\frac{(n+2)[2n+4]}{(n+1)(n+2)^2(n+3)}$$
$$=\frac{2(n+2)^2}{(n+1)(n+2)^2(n+3)}=\frac{2}{(n+1)(n+3)}$$

$$\frac{1}{(n+1)(n+2)}+\frac{1}{(n+2)(n+3)}$$
$$=\frac{n+3+n+1}{(n+1)(n+2)(n+3)}$$
$$=\frac{2n+4}{(n+1)(n+2)(n+3)}$$
$$=\frac{2}{(n+1)(n+3)}$$

Common student misconception (FP1)

- Because

$$\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{r=1}^{n} \frac{1}{r^2} = \frac{6}{n(n+1)(2n+1)}$$

Algebra – Bracket Expansion

- Many students have never taken on bracket expansion beyond those of the form (a + b)(c + d)
- Their experience of so much mathematics at best much less rich or at worst completely devoid of meaning for this.
- Particularly ideas associated with the binomial theorem/expansion/distribution which are everywhere in A-level mathematics.

The binomial expansion

- It's very easy to reduce this to a 'process' and, ultimately, this is probably what is required.
- However, it's worth spending some time reviewing the idea of a binomial coefficient, particularly if students are also studying S1, it's also worth reviewing the idea of expanding brackets

Expanding brackets

- An important point is that when we do this kind of algebra generally we are dealing with identities

$$2(x + 3) \equiv 2x + 6$$

$$(x - 2)(x - 3) \equiv x^2 - 5x + 6$$

- So again, the idea of substituting values into an expression is really, really important.

Expanding brackets

- Some questions to ask students:
  How many terms do you get when you expand each of the following?

$$(a + b + c)(e + f)$$

$$(a + b + c + d)(e + f + g)$$

$$(a + b)(c + d)(e + f)?$$

$$(a + b)(c + d)(e + f)(g + h)?$$

Expanding brackets

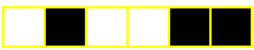
- Now let's consider

$$(x + y)(x + y)(x + y)(x + y)$$

- How many terms?
- How many different types of term?
- How many of each type?

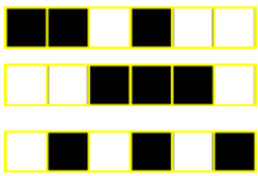
Binomial Coefficients

- How many different diagrams are there like this, where there are six boxes, three of which are black?



Binomial Coefficients

- Here are some of them but how many are there in total?



Binomial Coefficients

- We need a way of counting them. Imagine we are going to select the three black boxes one at a time
- What's wrong with the following? There are 6 options for the first black box, there are then 5 options for the second black box, then there are 4 options for the third one.
  So the answer is 6 x 5 x 4 = 120.

Binomial Coefficients

- Some of these will be the same

	1			2	3
	1			3	2
	2			1	3

	3			1	2
	2			3	1
	3			2	1

Binomial Coefficients

- So the number of repetitions is the same as the number of ways of arranging the numbers 1, 2 , 3 in a line.

1 2 3
1 3 2

2 1 3
2 3 1

3 1 2
3 2 1

Binomial Coefficients

- So the number of distinct diagrams is

$$\frac{6\times5\times4}{3\times2\times1}$$

- How many distinct diagrams like this? 9 boxes, 5 of which are black.

Binomial Coefficients

- How many diagrams are there of n boxes with r of them coloured black?
- Answer

$$\frac{n\times(n-1)\times(n-2)\times\ldots\times(n-r+1)}{r\times(r-1)\times(r-2)\times\ldots\times2\times1}$$

- I think this is a bit more transparent than the usual version of this formula

Back to the binomial expansion

- (x + y)(x +y)(x + y)(x + y)

How many x^3y terms will there be? This is the same number of ways of choosing the three brackets that the x's will come from. Imagine the terms in boxes:

(x + y)	(x + y)	(x + y)	(x + y)

What does this approach mean for students?

- If students understand this it will

  a) live with them when they are doing binomial expansion questions and make it more meaningful

  b) help them with their algebra in general because they've looked at expanding brackets in a deeper way

  c) prepare them better for generalisations of the binomial expansion to come later

Pascal's triangle and the binomial theorem

- The additive property of Pascal's triangle becomes clear when multiplication is set out in a formal way

$$\begin{array}{r} a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 \\ a + b \\ \hline a^5 + 4a^4b + 6a^3b^2 + 4a^2b^3 + ab^4 \quad\quad \\ a^4b + 4a^3b^2 + 6a^2b^3 + 4ab^4 + b^5 \\ \hline a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5 \end{array}$$

Extending Pascal's Triangle

- Newton had the remarkable insight that the binomial theorem is not only true for positive integers but also for negative and fractional values of n.
- Lines corresponding to negative integers can added to Pascal's triangle

Extending Pascal's Triangle

1	-2	3	-4	5	-6
1	-1	1	-1	1	-1
1	0	0	0	0	0
1	1	0	0	0	0
1	2	1	0	0	0
1	3	3	1	0	0
1	4	6	4	1	0
1	5	10	10	5	1

$$\frac{1}{1+x} = (1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - $$

$$\frac{1}{(1+x)^2} = (1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + 5x^4 - $$

When is too much algebra?

- Sometimes algebra is guilty of making simple concepts look more difficult than they really are.
- Addition and multiplication are much more intuitive concepts than subtraction and division.
- For example, the former are commutative whereas the latter are non-commutative.
- In general if we can work with addition and multiplication things will be easier for us.

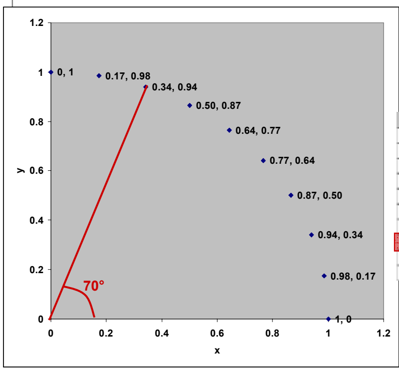
Approaches to trigonometry

- Usually trigonometry is introduced as being concerned with the ratio of side lengths in right angled triangles
- Ratio is a difficult concept algebraically and an approach based on similar triangles and enlargement might be better…
- Sometimes over-use of algebra is not the best approach

Sin and Cos

- Introduce them by getting students to see the output for various values of θ.

θ	cosθ	sinθ
0	1	0
10	0.98	0.17
20	0.94	0.34
30	0.87	0.50
40	0.77	0.64
50	0.64	0.77
60	0.50	0.87
70	0.34	0.94
80	0.17	0.98
90	0	1



θ	cosθ	sinθ
0	1	0
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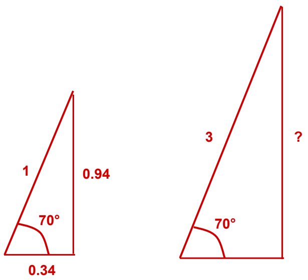
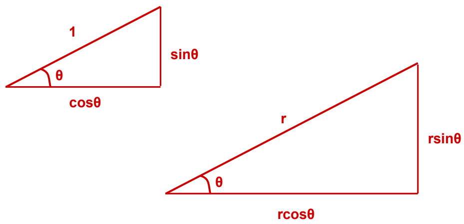
40	0.77	0.64
50	0.64	0.77
60	0.50	0.87
70	0.34	0.94
80	0.17	0.98
90	0	1

Doing problems using an enlargement technique



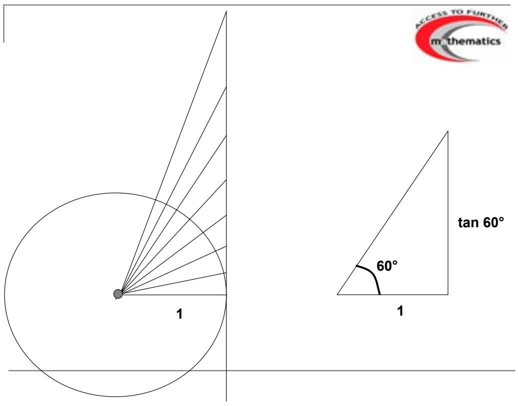
This then follows very naturally….



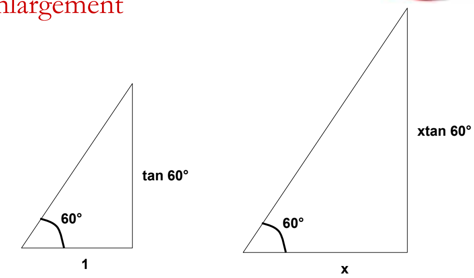
Contrast the approaches

Tan and Gradient

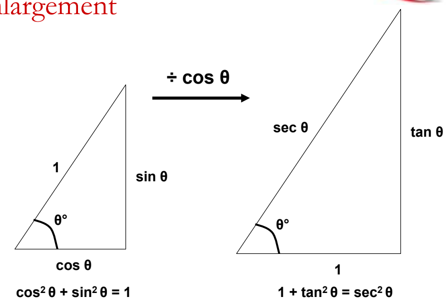
- Tan often gets overlooked in A-level mathematics and is frequently dealt with as sin/cos.
- Giving tan attention in its own right and looking at the connection to gradient can pay dividends.



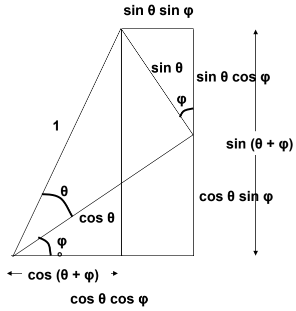
Defining tan as a length using enlargement



Defining tan as a length using enlargement



Geometric proofs



When don't you need algebra at all?

When it makes things look more confusing than they really are and can increase the probability of mistakes in application.

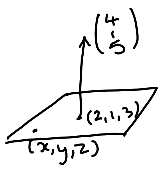
- Gradient
- Vector from pt A to pt B

Introducing a higher level concept

- The cartesian equation of a plane
- A good strategy is to use a simple numerical example that can act as an indicator for the algebraic steps that need to be taken.

Introducing concepts numerically

- A good example of this would be the cartesian equation of a plane.

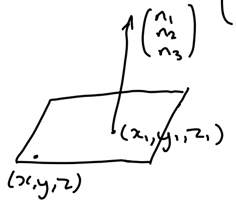


$$\begin{pmatrix}4\\1\\5\end{pmatrix}\cdot\begin{pmatrix}x-2\\y-1\\z-3\end{pmatrix}=0$$

$$\Rightarrow 4x+y+5z=24$$

Compared to…

$$\begin{pmatrix}n_1\\n_2\\n_3\end{pmatrix}\cdot\begin{pmatrix}x-x_1\\y-y_1\\z-z_1\end{pmatrix}=0$$



$$n_1x+n_2y+n_3z$$
$$=n_1x_1+n_2y_1+n_3z_3$$

Conclusion

- To students, algebra often seems to lack meaning and purpose
- Much current algebra teaching a resources encourage rote learning aimed at an instrumental understanding
- Understanding and fluency in performing operations with number is an essential prerequisite for success in algebra

Conclusion

- Students often interpret letter erroneously as objects.
- Misconceptions need to confronted and discussed.
- Dual role of expressions as a instruction and as an object which can be manipulated in its own right.

Conclusion

- Fluency in the skills of manipulating algebraic expressions is essential and requires practice through frequent repetition
- Don’t necessarily need ‘real world contexts’, surprising number properties and puzzles can be just as real to students
- The interplay between generalising and specialising is an important element in making sense of new ideas

Conclusion

- Algebraic topics often get isolated from other topics, it’s important to make connections
- Technology can help a lot but fluency in operating with simple algebraic expressions will continue to be an essential component in understanding and using algebra effectively.