Six Gems Every Teacher of Further Pure Should Know

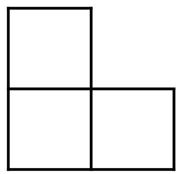
Chris Saker

Gem 1

Helping students to really understand how induction works

A Fun Proof by Induction

- Show that, for $n \geq 1$, a 2^n x 2^n grid can be covered by L shaped tiles: -



in such a way that only once cell is left uncovered and we can choose that cell to be any one we want.

A Fun Proof by Induction

- Base Step – the result is obvious when *n* = 1

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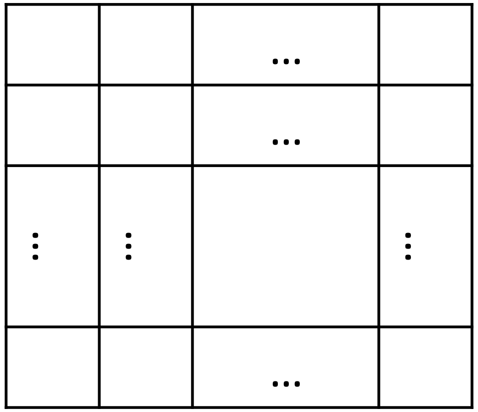
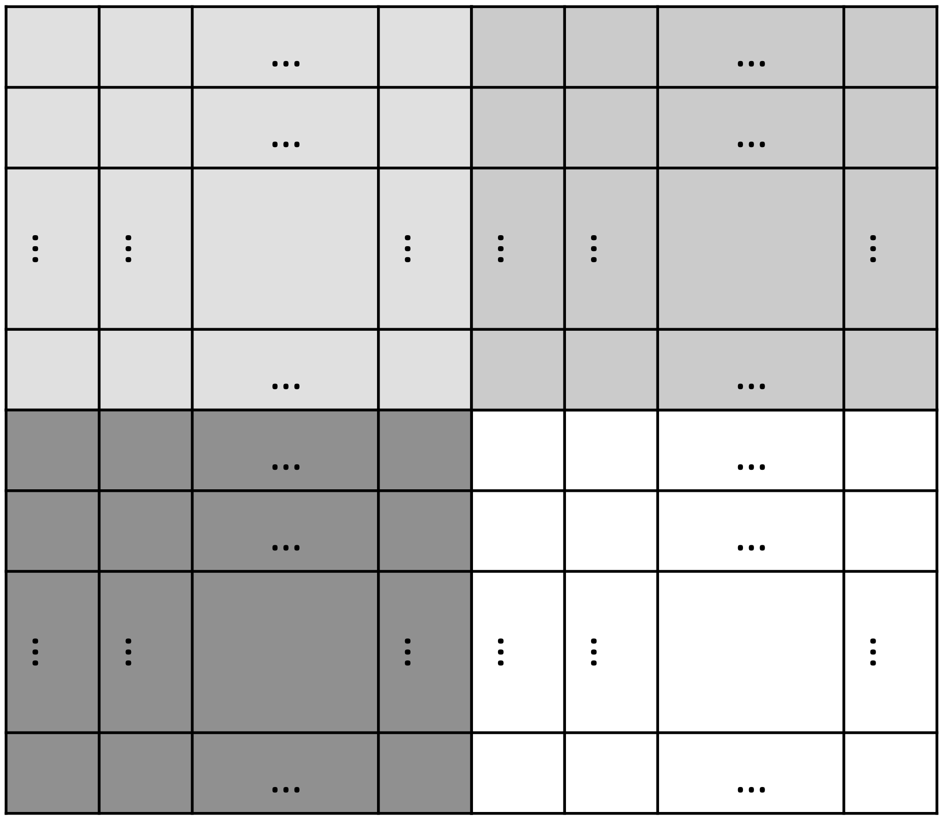
A Fun Proof by Induction

- Next assume there exists a *k* such that the result holds for a 2^k *x* 2^k grid: -



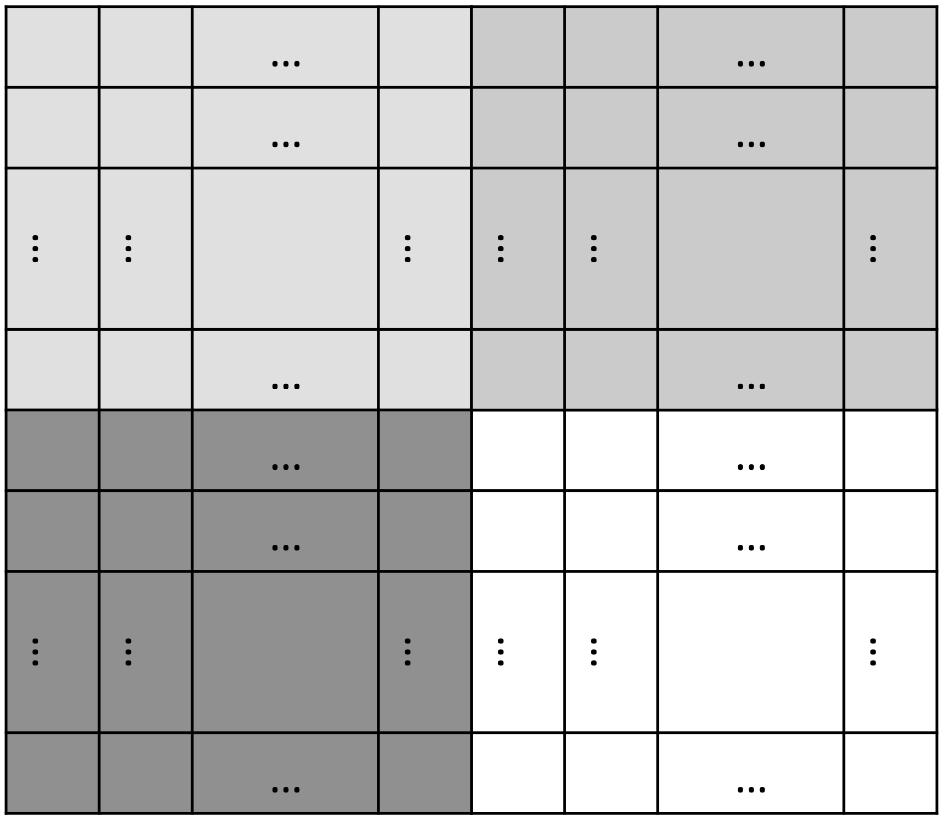
A Fun Proof by Induction

- …and show how it works when $n = k+1$



A Fun Proof by Induction

- (which checks their understanding of powers)



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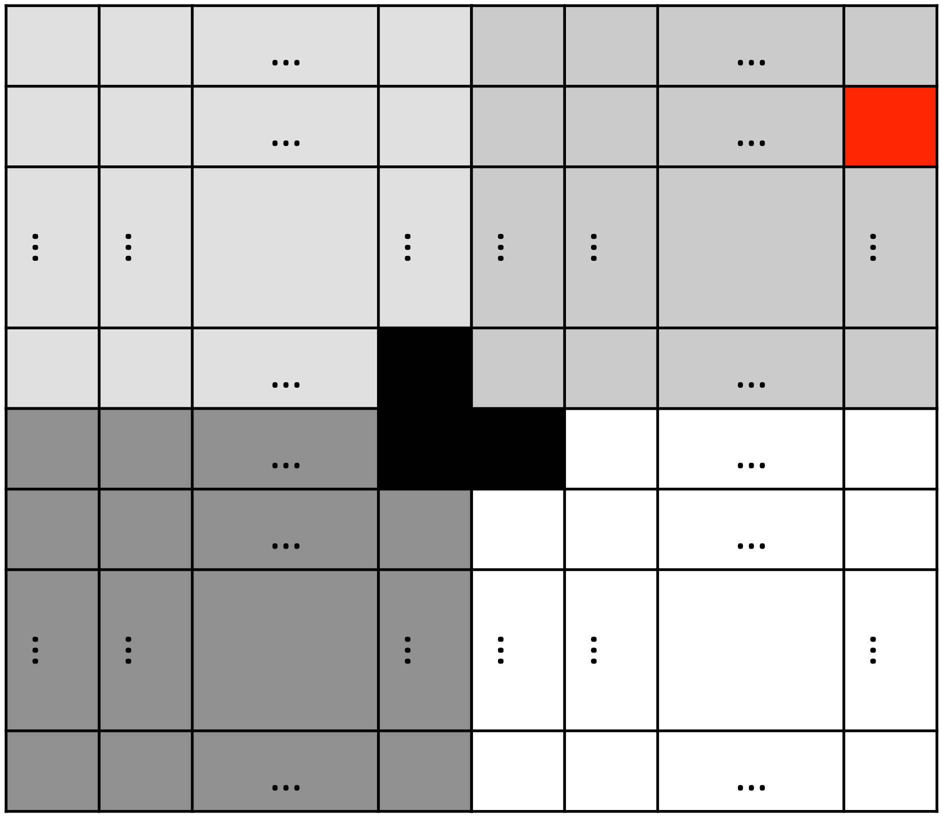
- …and show how it works when *n* = *k*+1



Gem 2

Inverse of a Matrix Product

By algebra

And by transformations

It is usual to demonstrate the following algebraically: -

- Assume A has two inverses, B and C, that is: -

$$AB = I = BA \text{ and } AC = I = CA$$

- It follows that: -

$$B = IB = CAB = CI = C$$

and therefore matrix inverses are unique.

- Next observe that: -

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$$

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But maybe is is easier to visualise via transformations…

- For example, given a point (*x*,*y*) first apply a rotation of 90^{o} anticlockwise followed by a reflection in the *x*-axis.

- Now ask which way you have to apply the inverse transformations to get back to their start point.

- This is (clearly) not a proof, but it does hopefully allow students to visualise why things here are the way they are!

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Gem 3

Strong Induction

Strong verses Weak Induction

- Some students seem to think that when doing an induction we are assuming that every thing holds true up to the k^{th} case as opposed the reality where we are making no assumption at all about the validity of the 2^{nd} – $(k\text{-}1)^{th}$ cases
- Without knowing it they are making the (stronger) assumption needed for strong induction so why not show them a strong induction to help them see the difference…

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The Fundamental Theorem of Arithmatic

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1) Base Case – *n* = 2:

As 2 is a prime the base case is ok

The Fundamental Theorem of Arithmatic

Proof

2) Inductive Hypothesis:

Assume the result holds true for all integers

$$n = 2, 3, \ldots, k$$

(so we have a stronger assumption than normal)

The Fundamental Theorem of Arithmatic

Proof

3) Induction Step – use the inductive hypothesis to show the result holds for *n* = *k*+1: -

When *n* = *k*+1 there here are two cases to consider, either: -

- *k*+1 is prime and we do not have to do anything, or
- *k*+1 is not a prime in which case there exist positive integers *a* and b strictly less than *k* such that *k*+1 = *ab*. Now, by the Inductive Hypothesis, both *a* and *b* have their prime factorizations and thus so does *k*+1.

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3) Induction Step – use the inductive hypothesis to show the result holds for *n* = *k*+1: -

When $n = k+1$ there here are two cases to consider, either: -

- $k+1$ is prime and we do not have to do anything, or
- $k+1$ is not a prime in which case there exist positive integers a and b strictly less than k such that $k+1 = ab$. Now, by the Inductive Hypothesis, both a and b have their prime factorizations and thus so does $k+1$.

This, with the exception of uniqueness, completes the proof.

But what about uniqueness?

You may not want to run through this but it can be good to get students to think about it…

Suppose $n = p_1 p_2 \ldots p_k = q_1 q_2 \ldots q_l$, where $k \leq l$, all the p_i's and q_j's are prime and the factors are written a monotonically increasing order.

Now, clearly p_1 divides $n = q_1 q_2 \ldots q_l$ and so it also divides some q_j and there is a natural number b such that $q_j = bp_1$.

Since q_j and p_1 are prime it follows that $b = 1$ and hence $p_1 = q_j \geq q_1$.

Arguing similarly in the other direction we can show $q_1 = p_i \geq p_1$. Thus, $p_1 = q_1$.

Next cancel $p_1 = q_1$ from the two factorisations of n and repeat the process.

Eventually we will get to $1 = q_{k+1} q_{k+2} \ldots q_l$ and we conclude that $k = l$ and the factorisations were in fact the same.

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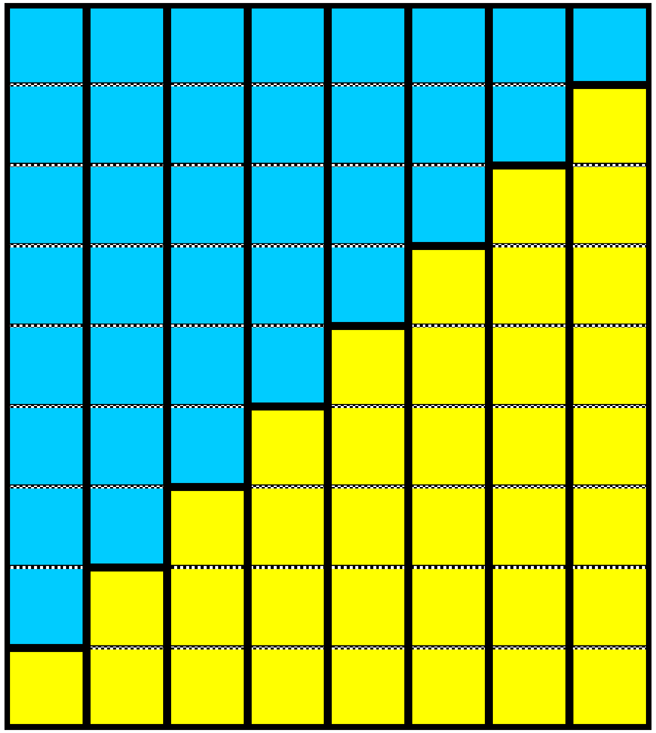
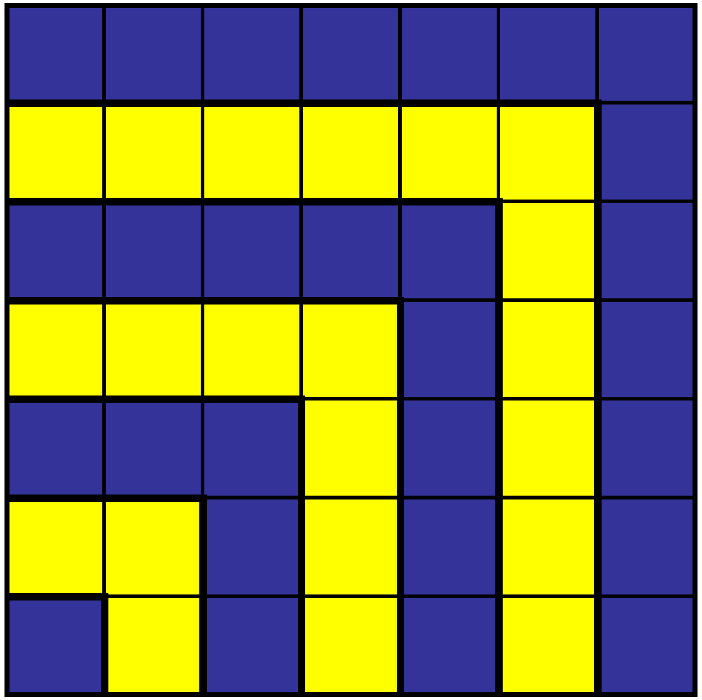
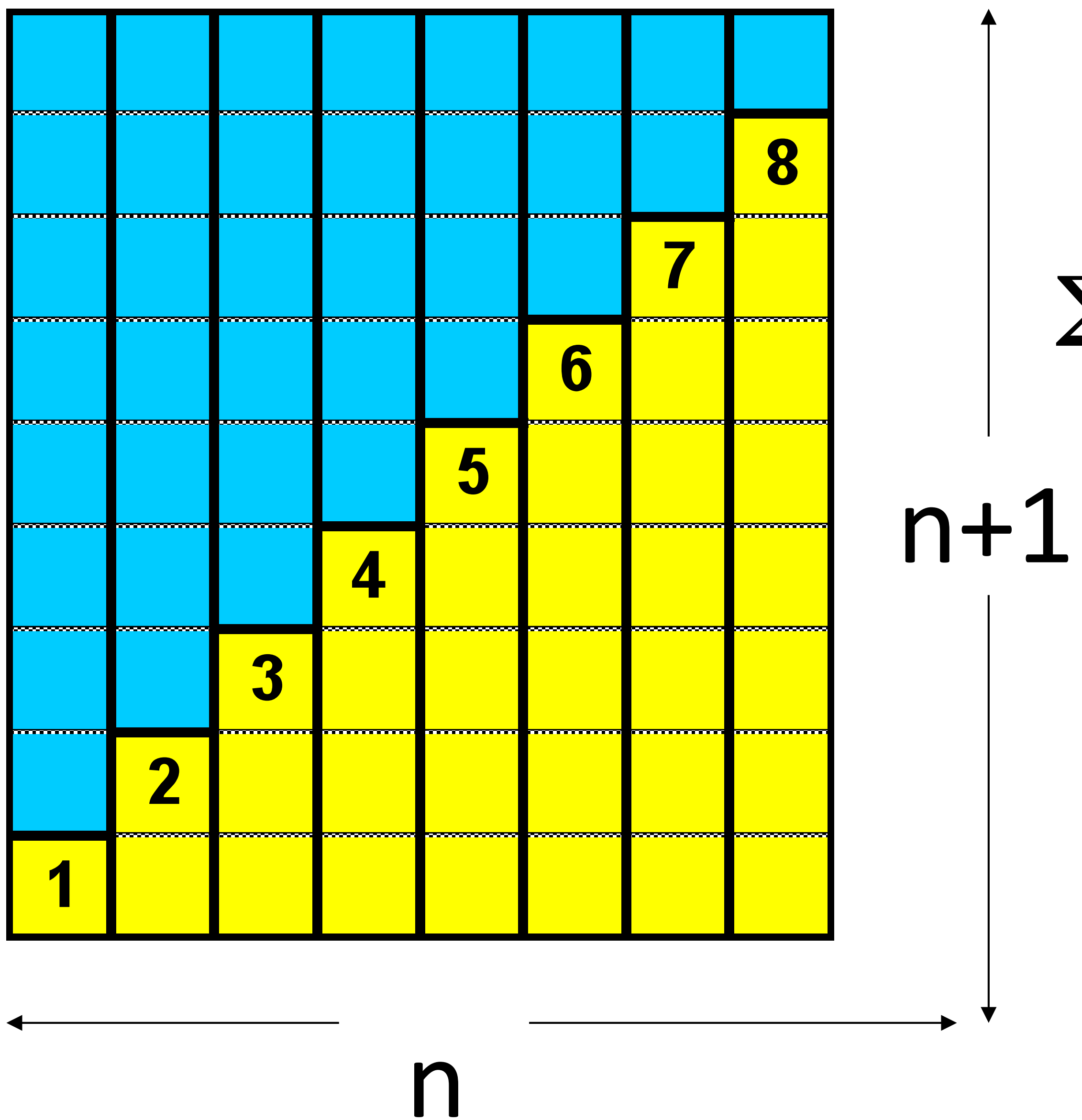
Gem 4

Proofs by Picture

Proof by pictures….

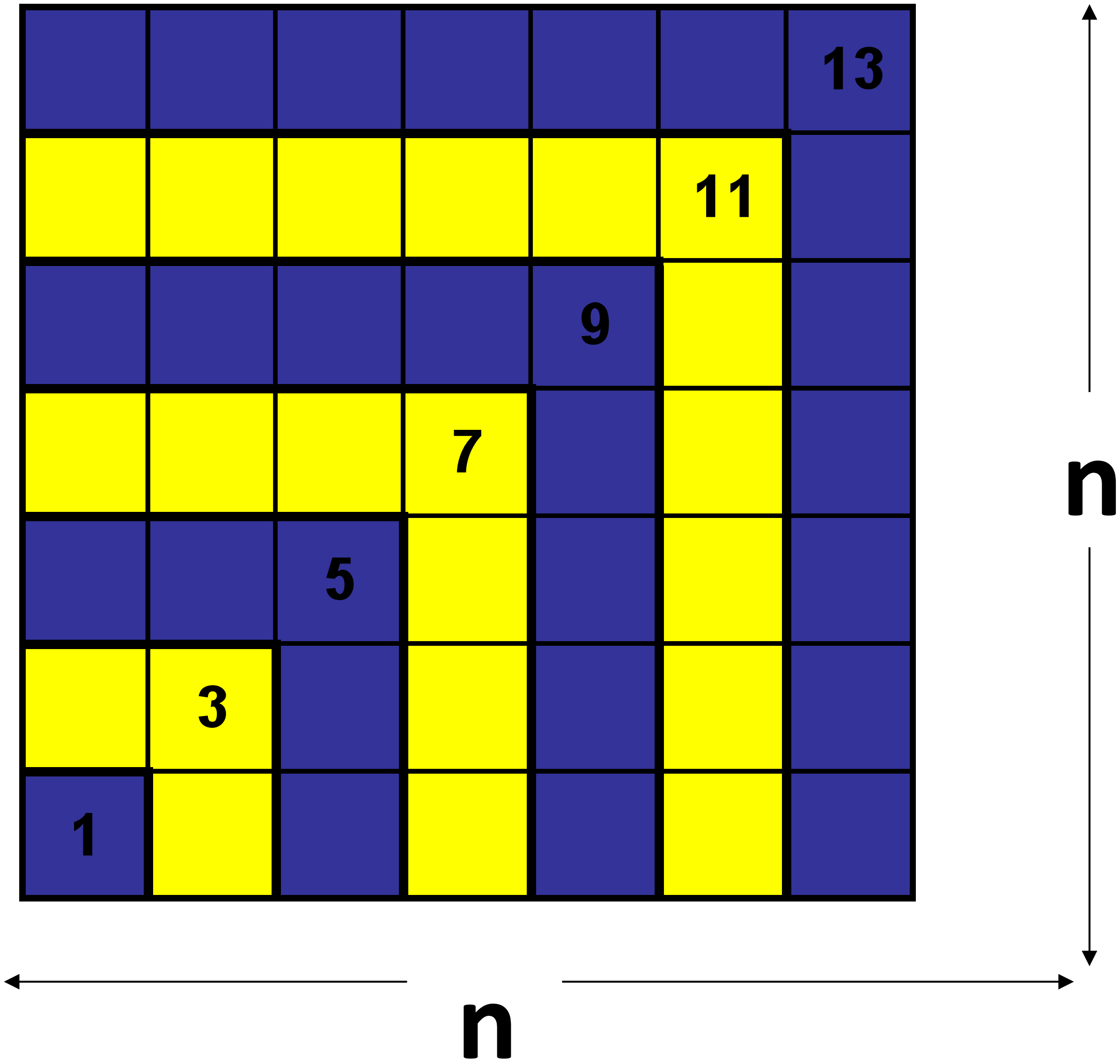


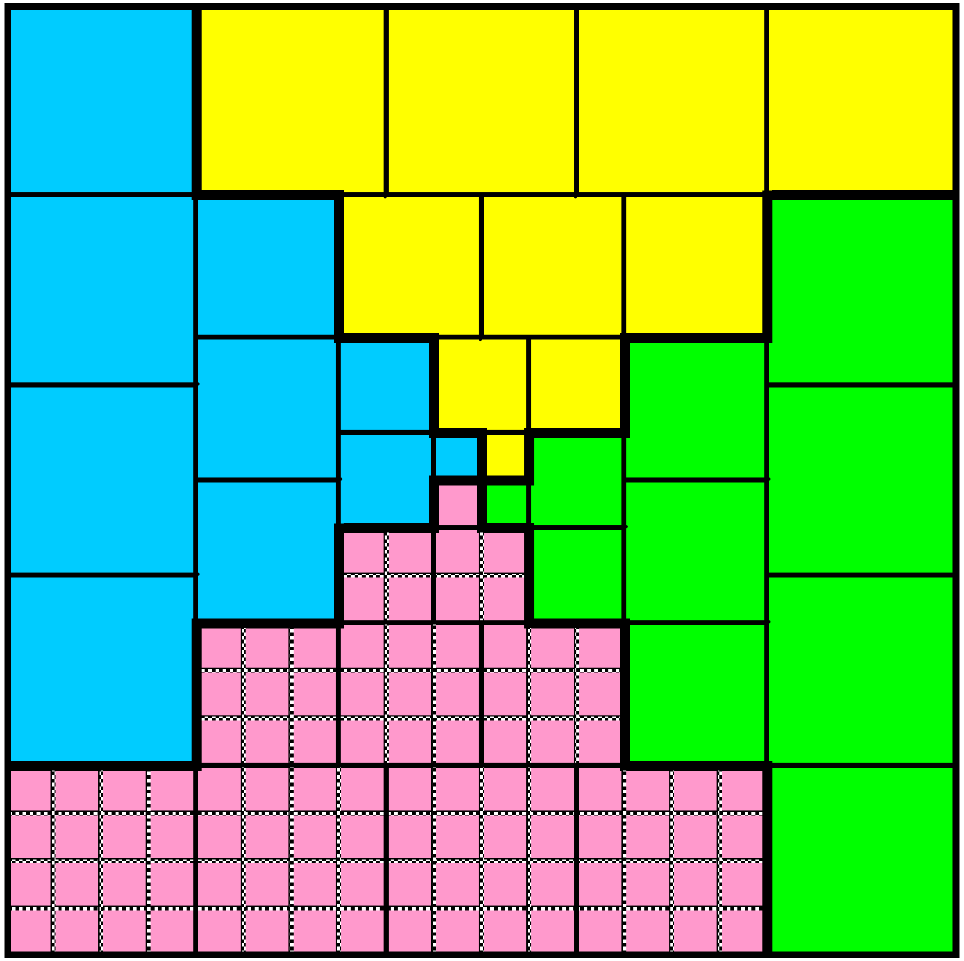


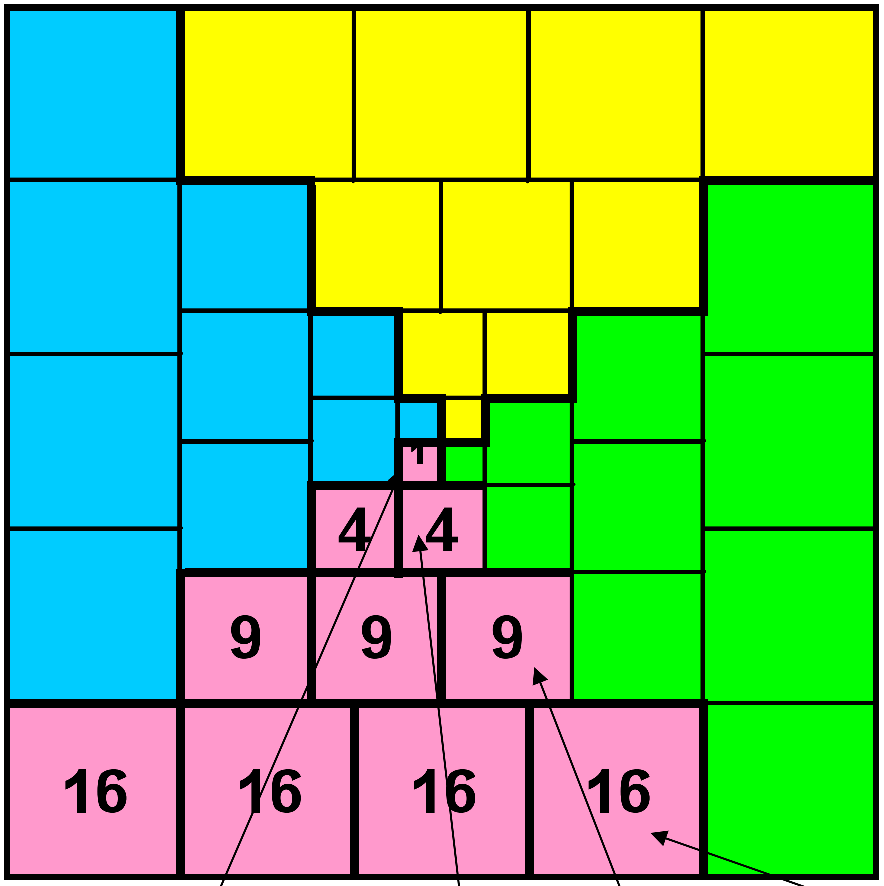


$$\Sigma r = \frac{1}{2} n(n+1)$$

$$\Sigma(2r-1)=n^2$$

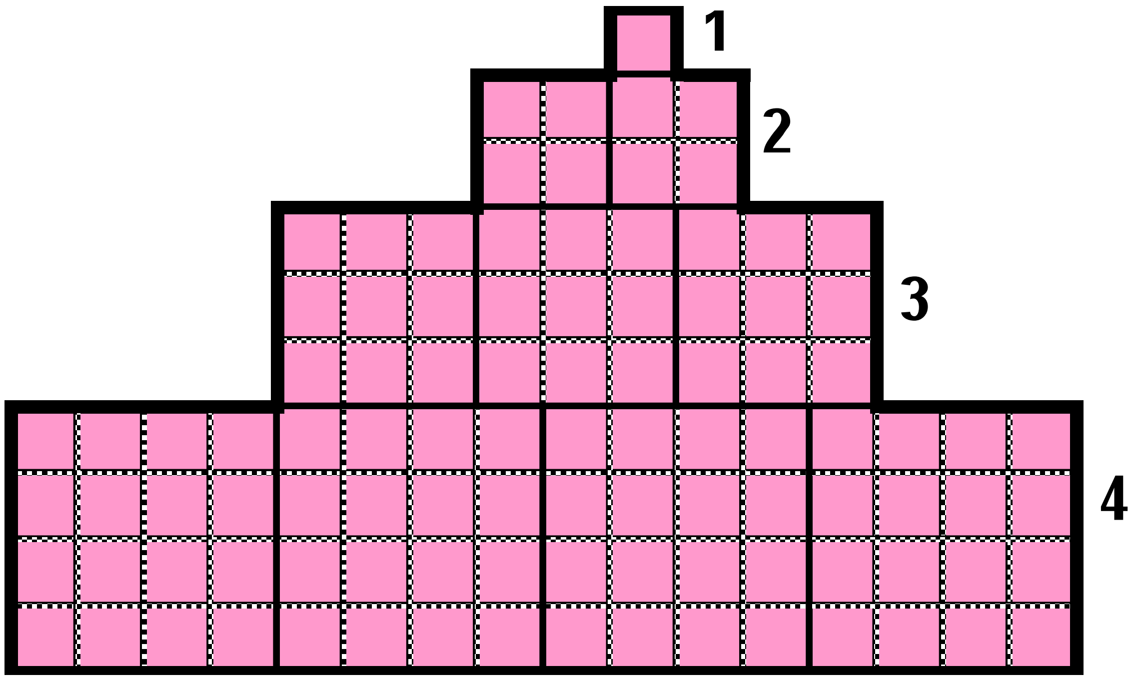






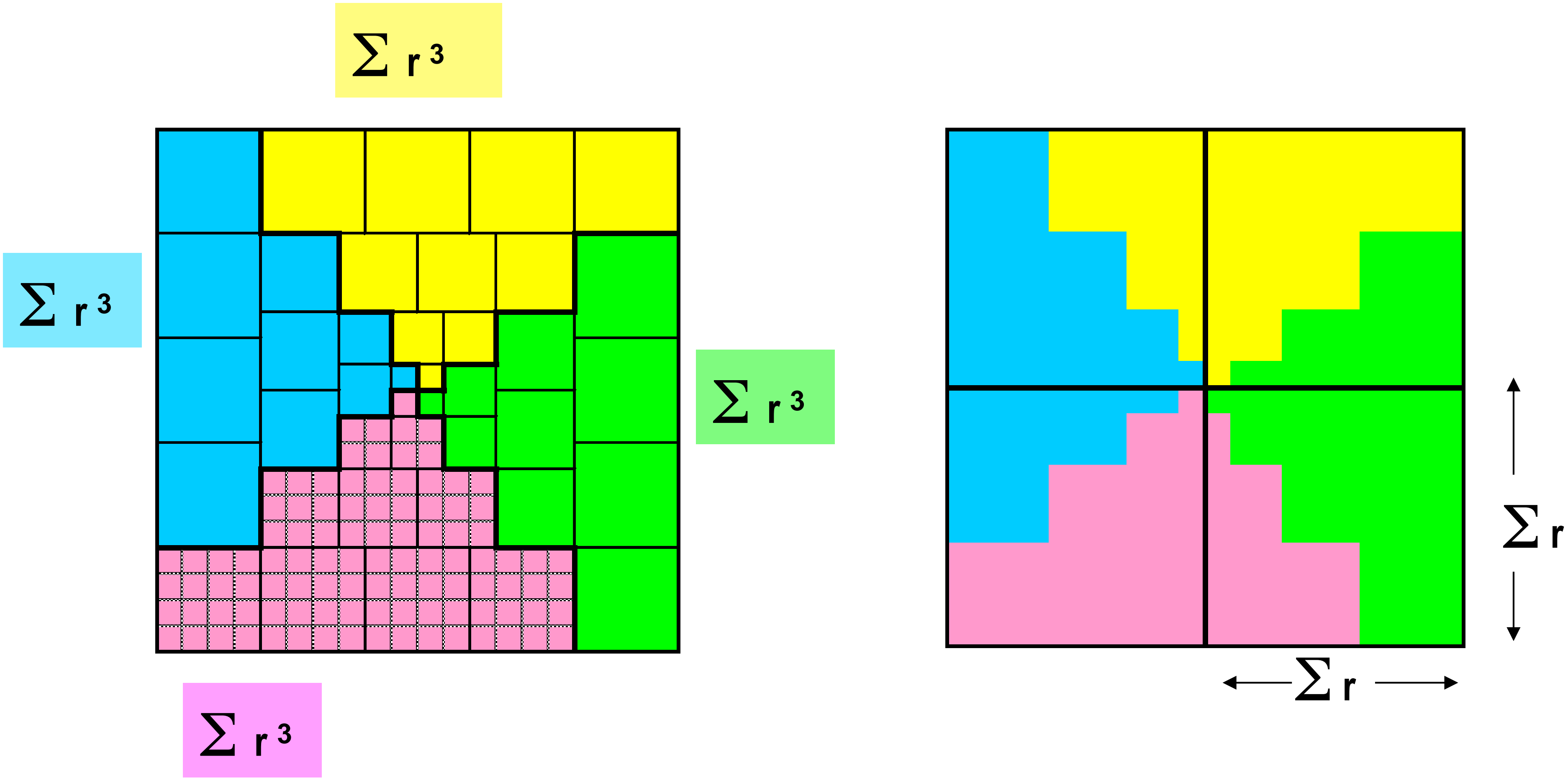
$$(1 \times \mathbf{1^2}) + (2 \times \mathbf{2^2}) + (3 \times \mathbf{3^2}) + (4 \times \mathbf{4^2}) + \ldots.(n \times \mathbf{n^2})$$

$$= \Sigma(r \times \mathbf{r^2}) = \Sigma\ \mathbf{r^3}$$



1 + 2 + 3 + 4 + ….+ n

$= \sum r$



$$\Sigma r^3 = (\Sigma r)^2$$

$$1^3 + 2^3 + 3^3 + 4^3 + \ldots + n^3 = (1 + 2 + 3 + 4 + \ldots + n)^2$$

Gem 5

Other Applications of Induction

Links to Decision 1

There are many proofs of the fact that the Euler Characteristic: -

$$\#(V) - \#(E) + \#(F)$$

of a simple connected planar graph G is 2.

We will now prove it via induction on the number of edges.

Base Case

If #(E) = 0 or #(E) = 1 then as we know G is connected it is easy to verify that

$$\#(V) - \#(E) + \#(F) = 2$$

Inductive Hypothesis

Let us suppose that the result holds for any graphs with #(E) ≤ k.

Inductive Step

Consider a graph where #(E) = k+1.

We are going to investigate what happens when we remove an edge from G.

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Inductive Step

Consider a graph where $\#(E) = k+1$.

We are going to investigate what happens when we remove an edge from G.

Inductive Step

There are two cases we need to consider here: -

Case 1: The edge we remove disconnects the graph

If the removal of an edge disconnects the graph then we are left with two graphs, both with $\#(E) \leq k$ and so each graph individually has Euler Characteristic 2.

Now, if we label the two components of *G* after the removal of the edge 1 and 2 we have: -

$$\#(V_1) - \#(E_1) + \#(F_1) = 2$$

$$\#(V_2) - \#(E_2) + \#(F_2) = 2$$

Inductive Step

There are two cases we need to consider here: -

Case 1: The edge we remove disconnects the graph

So, for our graph *G* we have: -

$$\#(V) = \#(V_1) + \#(V_2)$$

$$\#(E) = \#(E_1) + \#(E_2) + 1$$

(as we removed an edge)

$$\#(F) = \#(F_1) + \#(F_2) - 1$$

(as we are double counting the outside face)

Inductive Step

There are two cases we need to consider here: -

Case 1: The edge we remove disconnects the graph

Now: -

$$
\begin{aligned}
4 &= (\#(V_1) - \#(E_1) + \#(F_1)) + (\#(V_2) - \#(E_2) + \#(F_2)) \\
&= (\#(V_1) + \#(V_2)) - (\#(E_1) + \#(E_2)) + (\#(F_1) + \#(F_2)) \\
&= \#(V) - (\#(E) - 1) + (\#(F) + 1) \\
&= \#(V) - \#(E) + \#(F) + 2
\end{aligned}
$$

and so: -

$$\#(V) - \#(E) + \#(F) = 2$$

Case 2: The edge we remove leaves the graph connected
This case is much easier, the edge we remove will cause two faces to become one and so applying the inductive hypothesis we have: -

$$\#(V) - (\#(E)-1) + (\#(F)-1) = 2$$

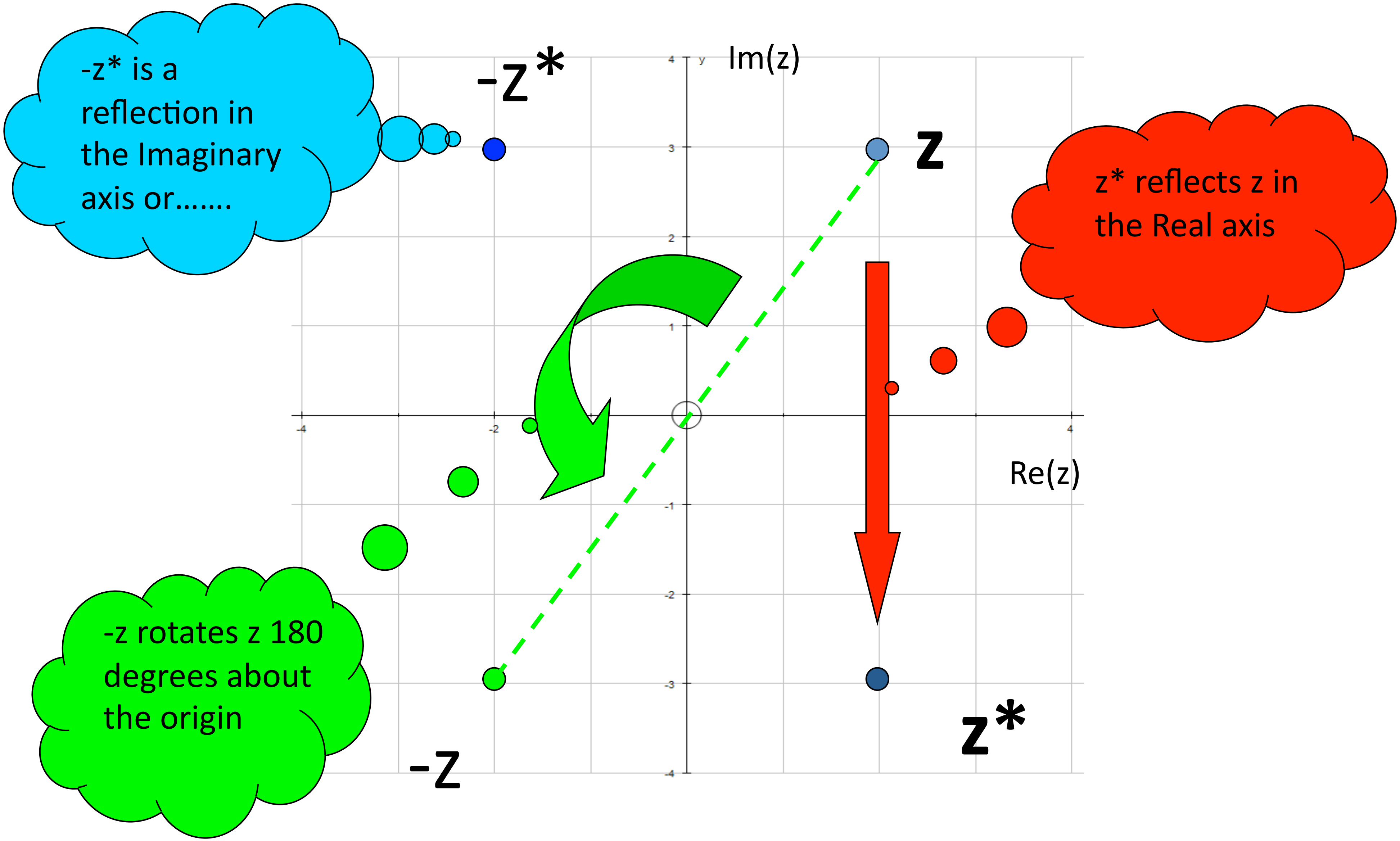
that is

$$\#(V) - \#(E) + \#(F) = 2.$$

Therefore, if the result is true for all graphs with at least *k* edges then it is also true for all graphs with *k*+1 edges and so the result is proved

Gem 6

Linking z* with -z


-z* is a reflection in the Imaginary axis or…….
-z*
Im(z)
z
z* reflects z in the Real axis
Re(z)
-z rotates z 180 degrees about the origin
-z
z*

Rotate then reflect

$$(-z)^* = -(z^*)$$

Reflect then rotate

Using $z = x + iy,$ show that

a rotation through 180 about the origin followed by

a reflection in the x axis

is the same as doing the reflection followed by the rotation:

Gem 7

Zombie Apocalypse?

I know DE has its own module with MEI but…

… this is quite good fun and shows a nice application of Differential Equations: -

http://www.mathstat.uottawa.ca/~rsmith/Zombies.pdf

The maths might be a bit too complicated but is does offer an interesting starting point for a discussion of applications of differential equations (Lotka-Voltera or spread of disease for example) which might be of interest to Further Mathematics students.

Thanks is owed to…

David Bedford
Sharon Tripconey
Steve Wall

Who all contributed ideas for this session

Any Questions?

Or are there any other Gems people would like to share?