Inversion – Geometry on its head…

**Bibliography:**

Geometry revisited	**Coxeter & Greitzer**	**MAA 1967**
Introduction to Geometry	**Coxeter**	**Wiley Classics 1961**
Algebraic Projective Geometry	**Semple & Kneebone**	**OUP 1952**
Complex Numbers in Geometry	**Yaglom**	**Academic Press 1968**
Electricity and Magnetism	**Jeans**	**CUP 1948**
Mathographics	**Dixon**	**Blackwell 1987**

A

Definition of Inversion

Two points P and P' are said to be inverses of each other with respect to a circle – centre O, radius k – if OPP' are collinear and OP.OP' = k^2.
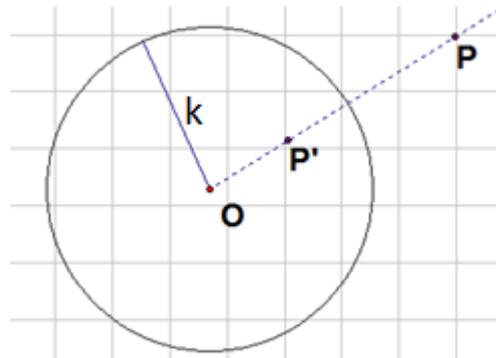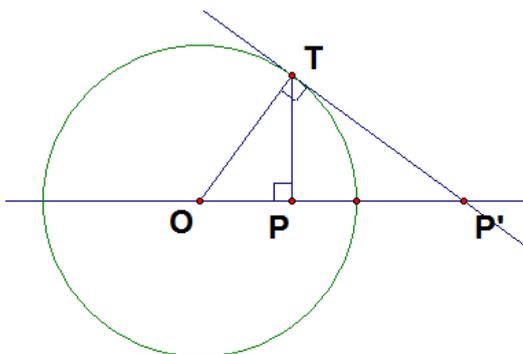
B

Construction of Inverse points:

1.   Can we use Vectors?

In terms of vectors, we want

        …

2.   Euclidean Approach

Construct the circle of inversion, centre O

Consider a typical point P (shown inside the circle of inversion, on the arbitrary axis OP)

Raise a line from P perpendicular to this axis, to cut the circle at T

The tangent at T cuts the axis at P', which is the inverse of P.   **Why?!**

If the original point is *outside* the circle of inversion, then we reverse the procedure, …

3. Alternative Geometer's SketchPad construction, using the idea of Intersecting Chords…

AP.PC = DP.PE … **= PT²** (or k²)
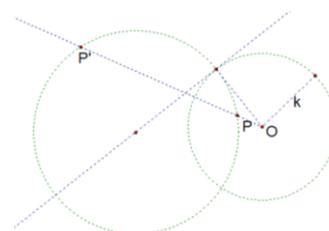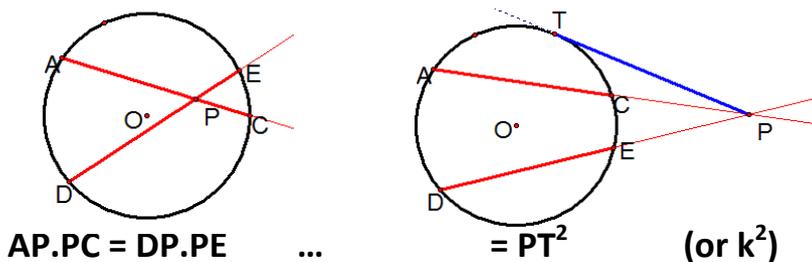
This looks awfully like the Inversion formula, where P is the centre of inversion, radius PT, and points A and C are inverses, as are D and E…

4. Using Complex Numbers:

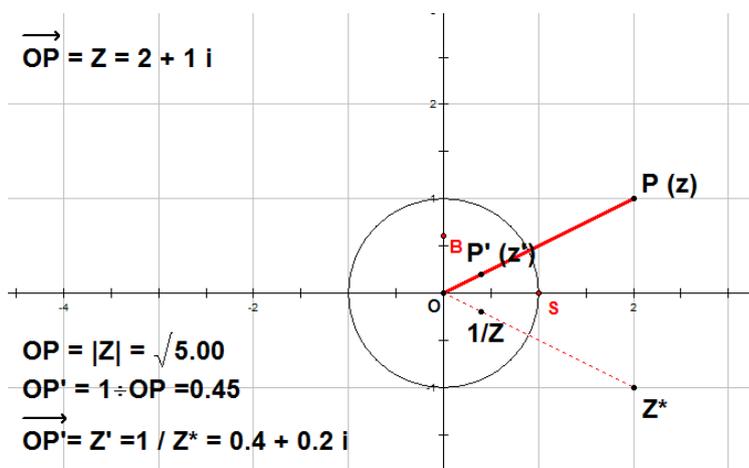With the usual notation, let's call z = a + bi, and its conjugate z* = a − bi.

To simplify the algebra, we'll take the centre of inversion to be O, and the radius k=1.

We want the inverse of z, which we'll call z', where

$$z' = \underline{\quad}$$

But z.z* = …

So, z' = …

$\overrightarrow{OP} = Z = 2 + 1\,i$

P (z)

B P' (z')

1/Z

O

S

$OP = |Z| = \sqrt{5.00}$

$OP' = 1 \div OP = 0.45$

$\overrightarrow{OP'} = Z' = 1\,/\,Z* = 0.4 + 0.2\,i$

Z*

This isomorphism between scaling magnitudes and division of complex numbers makes inversion an algebraic operation rather than a geometric one. Two strings to our bow!

5. Using Co-ordinate systems:

Polars are simplest… $r' = \underline{\quad}$; θ' = θ

Cartesian $\underline{\quad}$; $\underline{\quad}$

C

Some Preliminary Investigations

Use pencil, ruler and polar graph paper to 'invert' a few basic shapes, using a few points to get the idea of how the shapes distort as they are inverted…

A line – perhaps intersecting the circle of inversion, or not…

A square or triangle – equilateral, or not…

A circle – through the centre of inversion, or not…

Then use SketchPad (or Cabri etc) to do the same thing…

D   Some Euclidean proofs of our findings…

We saw how the Intersecting chords idea suggested that as we trace out P, P' also follows the same circular locus - but this was for a rather specific circle ('orthogonal' to the circle of inversion). Here's a proof:

In the diagram, we have a circle and an external point O.

The 'Cyclic quad' tells us about the angles in the diagram, and hence that ΔOPQ ~ ΔOQ'P'.

Thus —— ——, and hence OP.OP' = OQ.OQ'.

In the limit, as P and P' tend to each other, this becomes OP.OP' = OQ.OQ' = k², as required.

This is the same relationship as required for P and P' (and Q and Q') to be inverses of each other in a circle, centre O

and radius k, as OP' = —— is the condition for Inversion.

i.e.   P and P' are inverse points, and the circle QPP'Q' is self-inverse (or 'invariant') under Inversion in the circle, centre O and with a radius tangent to the invariant circle.

But what about other (less special, not-invariant) circles? Do all circles invert to circles?

So now consider two circles, *their line of centres passing through some external point O.*

Then the circle closest to O will be intersected by OPP' and OQQ', as shown, at P'' and Q''.

Since O is a centre of enlargement for one circle onto the other, *ΔOP''Q'' ~ ΔOP'Q'*

So, —— ——   and we already know that *OP.OP' = OQ.OQ'* so now —— ——

This is the same as *OP.OP'' = OQ.OQ'' … = k²*, say, showing that P and Q are the inverses of P'' and Q'' with respect to a circle, centre O and radius k. There is nothing special about P or Q, so this inversion will be true for the whole circle on the right, which inverts into the whole circle on the left.

(Having established that P and Q are the inverses of P'' and Q'', we may find the radius of inversion, k, as the length of the tangent from O to the circle we can draw through the four points P, Q, Q'' and P'', as we did in the first case above)

Hence, we have established that any circle will invert to another circle.
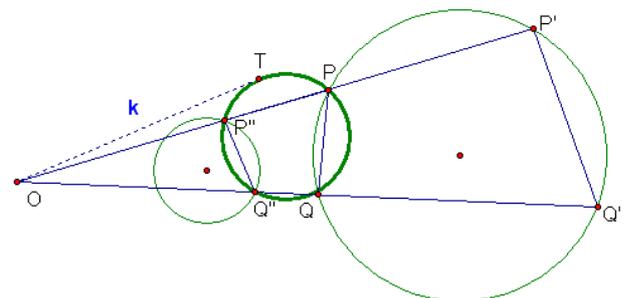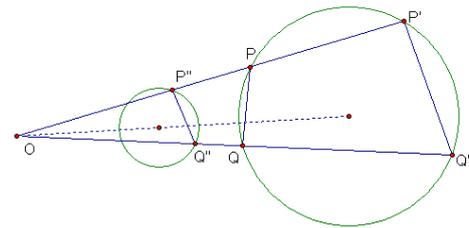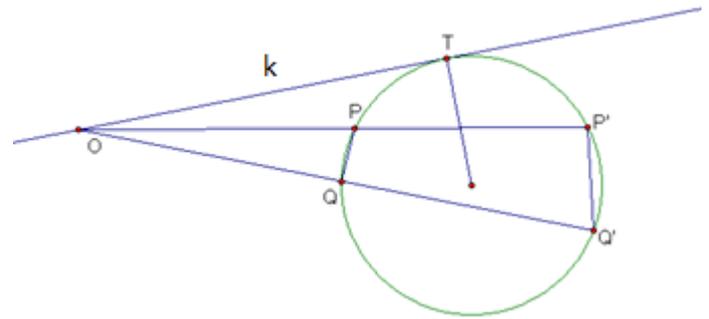
What about straight lines?

The other special case, however, is a circle that passes through the centre of Inversion, for then OP = 0, and OP' = — ,

which places the inverse of P at infinity. This turns out to be a circle of infinite radius, known to us finites as a straight line…

There are various proofs of this. Here's one:
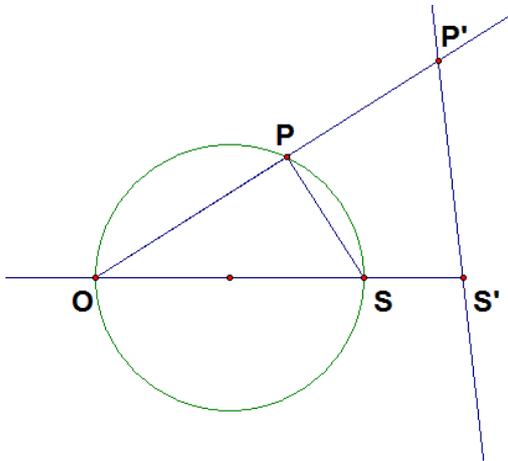
Consider a circle passing through the centre of inversion O.

Let the point S be diametrically opposite O. Extend the line OS as far as the inverse point S', where OS'.OS = k^2.

Now consider any point on the circle P, and its inverse P'.

The inversion relationship gives OP.OP' = OS.OS' = k^2, whatever the radius of inversion k may be,

which is the same as —   —

Since ΔOPS and ΔOS'P' share an angle at O, and the above ratios are the same, we deduce that the two triangles are similar, and hence that <OPS = <OS'P'.

But <OPS = 90° as OS is a diameter, and hence <OS'P' = 90°also.

This means that the inverse of any point on the circle, P, lies on a straight line perpendicular to OS.

So we have shown that the inverse of any circle through O is a straight line, and conversely that the inverse of any straight line is a circle through O.

[The converse consists in considering the point P' on the straight line perpendicular to OSS', and showing that <OPS = 90°, and hence that P describes a semicircle on OS as diameter]
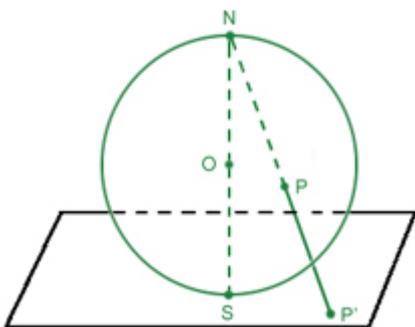
This very much reminds one of the projection of a circle (or sphere) onto a line (or plane) called a stereographic mapping. (Thanks to www.mathscareers.org.uk for this)

It works like this. Picture a glass sphere resting on a table, with a tiny laser pointer at the north pole. Shine the laser pointer through the sphere and you'll see two dots; one on the surface of the sphere and another on the table. This effectively means that a particular point on the sphere can be mapped to a particular point on the table without any overlapping - with just one exception. It's not possible to direct the laser pointer from the north pole to the table without passing through the sphere. Because no overlapping is allowed, the north pole can't be part of the projection.

What does the full projection look like? It's easiest to think about what happens to the equator, which is mapped to a circle with a radius twice that of the sphere's. Everything in the lower half of the sphere (the southern hemisphere) is mapped inside this circle, while everything in the upper half of the sphere (the northern hemisphere) is mapped outside.
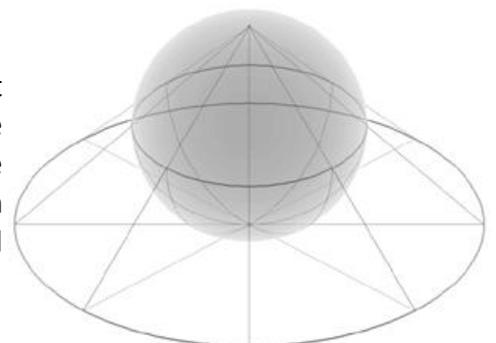
E Looking at Inversion with the help of Complex Numbers

To describe a circle in the Argand diagram, one can use the equation

i z.z* + b.z – b*.z* + ic = 0   ##

where z* is the conjugate of z, and b is a complex number defining the centre and c is a real number.

Let's see how it works.

Put z = x + iy, and b = p + iq, so that z* = x – iy and b* = p – iq.

Now ## becomes $i(x + iy)(x - iy) + (p + iq)(x + iy) - (x - iy)(p - iq) + ic = 0$

Several terms cancel: $i(x^2 + y^2) + (2ipy + 2iqx) + ic = 0$

which rearranges to $(x + q)^2 + (y + p)^2 = p^2 + q^2 - c$

So equation ## describes a circle in the complex plane, with centre (-q, -p) and radius $\sqrt{(p^2 + q^2 - c)}$

Now, **to invert this general circle** we can use the transformation z = —, and our ## equation

i.z.z* + b.z – b*.z* + i.c = 0   becomes

i. — — + b. — - b*. — + i.c = 0   which simplifies to   **i + b.z – b*.z* + i.c.z.z* = 0   @@**

which is clearly another circle, but with a different centre and radius (The co-ordinates of the centre are those of the original circle divided by c...)
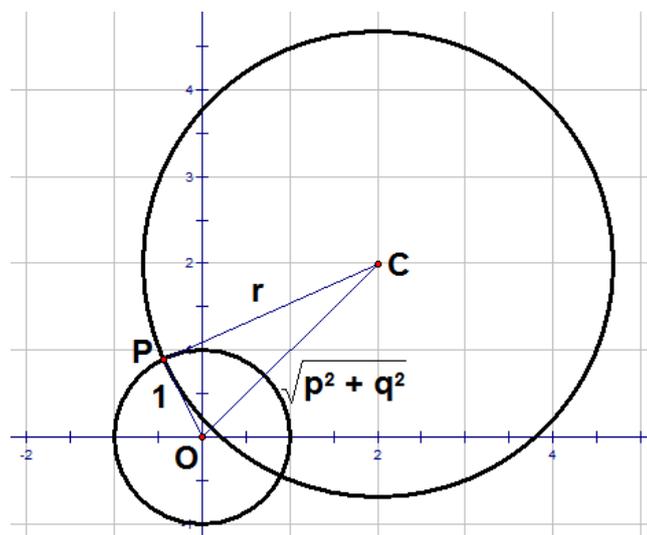
Invariant circles are easily recognised by the fact that we require c = 1 for ## to be the same as **@@.**

This leads us to the result that invariant circles have radius $\sqrt{(p^2 + q^2 - 1)}$, where (-q, -p) is the centre.

Claim: Invariant circles cut the circle of Inversion orthogonally.

For: Such circles are of the form $(x + q)^2 + (y + p)^2 = p^2 + q^2 - 1 = 0$

So centre is C(-q, -p), and radius r = $\sqrt{(p^2 + q^2 - 1)}$

Thus $r^2 = p^2 + q^2 - 1$ or $p^2 + q^2 = r^2 + 1$

Which, it is clear from the diagram (where q = p = -2), is the requirement that <OPC = 90°. So OP is ⌐to PC, which makes the 2 circles orthogonal.

So we have deduced that all invariant circles also intersect the circle of inversion orthogonally.

Circles that pass through the centre of the Inversion must have c = 0, so that z = 0 satisfies ##.

The inverted equation @@ then becomes **i + b.z – b*.z* = 0**   which can be written as

i + 2qix + 2piy = 0   or   qx + py + ½ = 0   – clearly a straight line.

i.e. Any circle that passes through the centre of Inversion inverts to a straight line, and vice versa.

In particular, any line through the centre of inversion (a 'circle of infinite radius') can be written as

b.z – b*.z* = 0   and inverts to the same line, as can be checked using the transformation z = ⎯
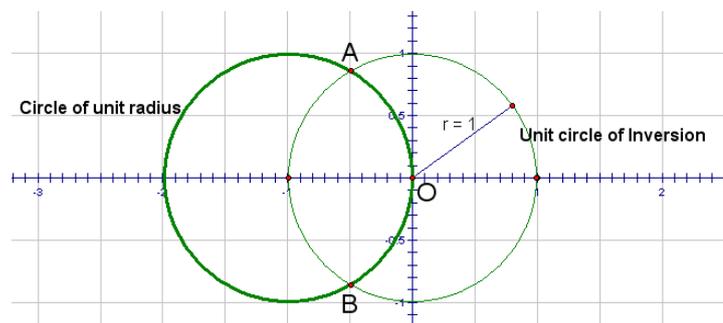
So certain lines are themselves invariant under inversion – all those through the centre of inversion.

F   Puzzle set

Puzzle 1

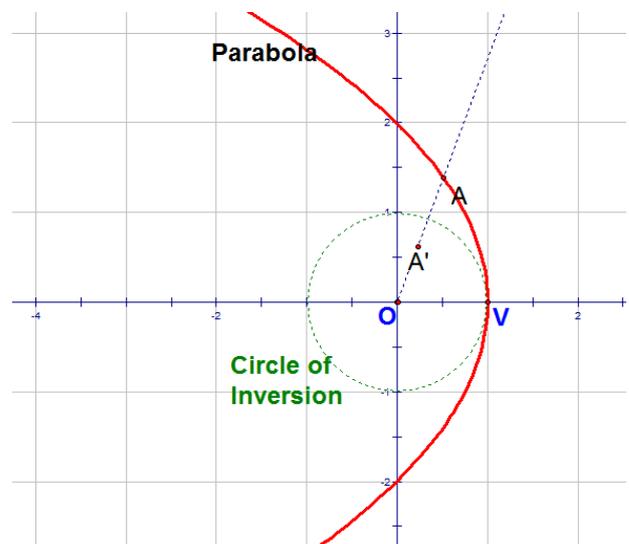A circle of radius 1 unit passes through the origin.

What is its inverse in the unit circle of inversion?

Puzzle 2

What special locus do you get when you invert a
parabola about its focal point?

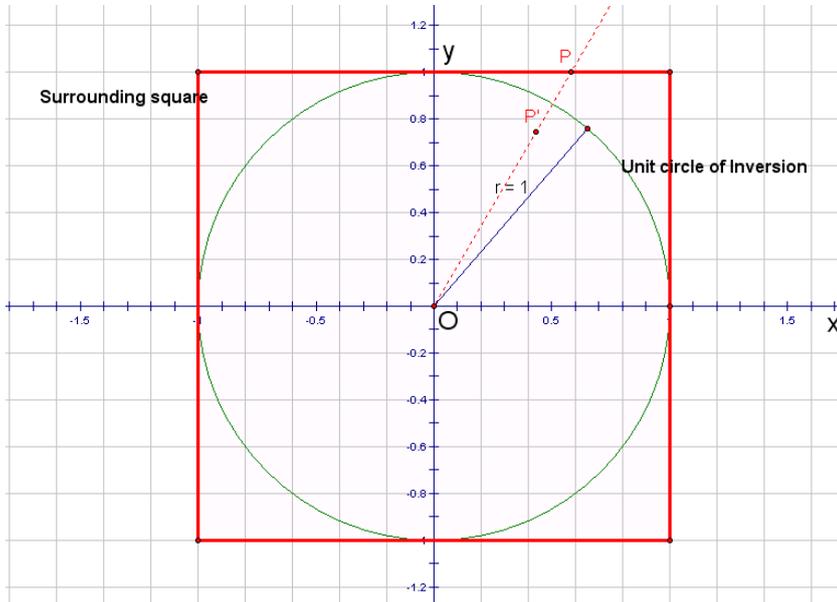(the Polar equation of this parabola is ⎯   )

Puzzle 3

Draw a square to exactly circumscribe the unit circle of inversion.

What is the inverse of this square?

[Hint: consider the sides as infinite straight lines…]

Puzzle 4

What are the co-ordinates of a typical point (x, y) in a circle of inversion of radius k, rather than 1?

Puzzle 5

Consider a circle with centre O, and any point P, either inside or outside the circle.

Join P to O, and construct the perpendicular diameter to OP. Label this AB (either sense).

Join A to P, and extend to the point where this line cuts the circle again at N.

Join B to N and extend to where it cuts OP extended. Label this point P'.

Problem: prove that P and P' are inverses w.r.t. the circle.

[Hint: look for similar triangles, calling the radius of the circle k]