Further Mathematics Support Programme

www.furthermaths.org.uk

the **Further Mathematics** Support Programme
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Members only

an introduction to groups

department for education and skills — creating opportunity, releasing potential, achieving excellence
National Centre for Excellence in the Teaching of Mathematics
MEI Mathematics in Education and Industry

Latin squares

A **Latin square** is an $n \times n$ table filled with n different symbols in such a way that each symbol occurs exactly once in each row and exactly once in each column.

1	2	3	4
2	3	4	1
3	4	1	2
4	1	2	3

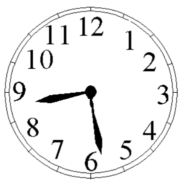
1	2	3	4
2	4	1	3
3	1	4	2
4	3	2	1

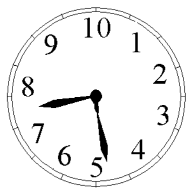
The great Swiss mathematician Leonhard Euler introduced the idea of latin squares in 1783. He used Latin characters as symbols, hence the name.

Uses:
error correcting codes
Experimental design

MEI 2010

Clock arithmetic





9 + 4 = 1 (mod 12)   9 + 4 = 3 (mod 10)

The number X (mod Y) is the remainder when X is divided by Y.

MEI 2010

Binary operations

Most operations take two elements of a set and combine them to give a definite result; such a rule of combination is called a **binary operation.**

MEI 2010

Modular arithmetic

What are the similarities and differences?

+	0	1	2	3	4
0					
1			3		0
2					1
3					
4					3

X	0	1	2	3	4
0					
1			2		
2					3
3					
4					

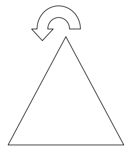
MEI 2010

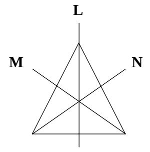
Modular arithmetic – mod 5

$+_5$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

X_5	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

Symmetry: equilateral triangle





Rotations

I 0º (or 360º)

R_1 120º anticlockwise

R_2 240º anticlockwise

Reflections

L reflection in line L

M reflection in line M

N reflection in line N

Groups

	first						
	*****	**I**	**R_1**	**R_2**	**L**	**M**	**N**
	I			R_2			
	R_1						
second	**R_2**			R_1			
	L				I		
	M		L				
	N					R_1	

Groups

	first						
	*****	**I**	**R_1**	**R_2**	**L**	**M**	**N**
	I	I	R_1	R_2	L	M	N
	R_1	R_1	R_2	I	M	N	L
second	**R_2**	R_2	I	R_1	N	L	M
	L	L	N	M	I	R_2	R_1
	M	M	L	N	R_1	I	R_2
	N	N	M	L	R_2	R_1	I

Group axioms

A group (G, *) is a non empty set G with a binary operation * such that

- * is closed in G
  a*b ∈G for all a,b ∈G
- There is an identity element *e* ∈G
  such that a*e = e*a = a for all a ∈G
- Inverses: for all *a* ∈G, there is an element a^{-1} such that $a * a^{-1} = e$
- Associativity:
  a(b*c) = (a*b)*c* for all *a,b,c* ∈G

Groups

	first						
	*****	**I**	**R_1**	**R_2**	**L**	**M**	**N**
	I	I	R_1	R_2	L	M	N
	R_1	R_1	R_2	I	M	N	L
second	**R_2**	R_2	I	R_1	N	L	M
	L	L	N	M	I	R_2	R_1
	M	M	L	N	R_1	I	R_2
	N	N	M	L	R_2	R_1	I

	first			
second	*	I	R_1	R_2
	I	I	R_1	R_2
	R_1	R_1	R_2	I
	R_2	R_2	I	R_1

There is one subgroup

Is it a group?

All integers under subtraction

No – subtraction is not associative

3 – (6 – 2) = -1
(3 – 6) – 2 = -5

Is it a group?

{1, 10} under multiplication modulo 11

X_{11}	1	10
1	1	10
10	10	1

yes

Is it a group?

All irrational numbers under multiplication

No – it is not closed

$\sqrt{2} \times \sqrt{2} = 2$, which is rational

The missing axiom

Commutativity:

a b* = *b* a* for all *a,b,* $\in$G

Is $+_5$ commutative? Yes

Is X_5 commutative? Yes

Is * commutative? No

Groups that are also commutative are called Abelian groups after the Norwegian mathematician Neils Abel

- The theory of groups started early in nineteenth century in connection with the solutions of algebraic equations.
- Originally a group was the set of all permutations of the roots of an algebraic equation which has the property that combination of any two of these permutations again belongs to the set.
- Later the idea was generalized to the concept of an abstract group. An abstract group is essentially the study of a set with an operation defined on it.
- Group theory has many useful applications both within and outside mathematics.
- Groups arise in a number of apparently unconnected subjects. In fact they appear in crystallography and quantum mechanics in geometry and topology, in analysis and algebra and even in biology.

A bit of history

Euler, Gauss, LaGrange, Ruffini, Abel Cauchy, Klein, Cayley

Evariste Galois

Born: 25 Oct 1811 in Bourg La Reine (near Paris), France
Died: 31 May 1832 in Paris, France

PLUS Maths

- http://plus.maths.org/issue48/package/index.html#intro
- A package of readings and resources for teachers
- http://www-history.mcs.st-andrews.ac.uk
- Find out about your favourite mathematicians

Here is the ultimate Sudoku variation — Binary Sudoku:

1	

There are 10 possible boards, each more exciting than the last.

0	**1**
1	**0**

(of course, that's the binary number 10)