Further Mathematics Support Programme

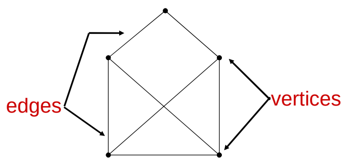
www.furthermaths.org.uk

Join the dots – Graph theory and gritting

Let Maths take you Further…

What is a graph?

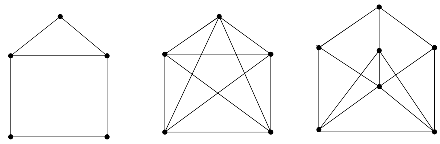
- A graph is a set of *points* called *vertices* (or nodes) connected by *lines* called *edges* (or arcs).
- In a graph a line from point A to point B is considered to be the same thing as a line from point *B* to point *A*.



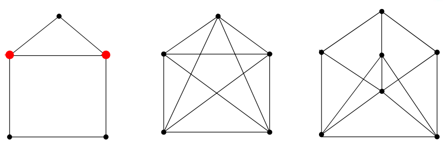
Graphs

Traversable graphs

Which of these graphs can be drawn without taking your pen off the paper or repeating any edges?



Graphs

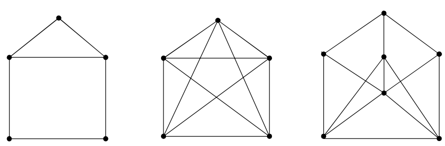


Yes –start and finish in different places	Yes- start and finish in the same place	No
Semi-Eulerian	**Eulerian**	

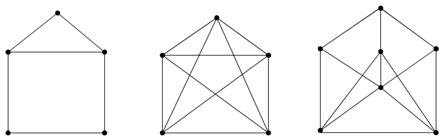
- What is significant about the results?

Graphs

- The **degree of a vertex** is the number of edges that meet at that vertex
- What do you notice about the number of odd vertices?
- Why?



What is the sum of the degrees of the vertices for each graph?



12 20 26

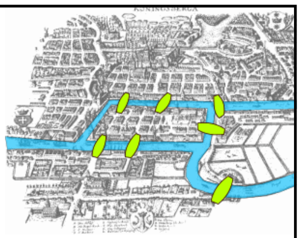
- What do you notice if you compare these totals to the number of edges in the graph?

Graphs

The sum of the degrees of all the vertices in a graph is twice the number of edges. This is more formally called the **Handshaking Theorem** and is written

$$\Sigma \deg v = 2e$$

Konigsberg bridges



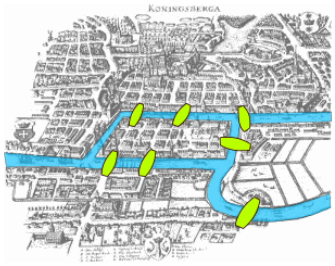
The **Königsberg bridges** is a famous mathematics problem inspired by an actual place and situation.

The city of Königsberg on the River Pregel in Prussia includes two large islands which were connected to each other and the mainland by seven bridges. The citizens of Königsberg allegedly walked about on Sundays trying to find a route that crosses each bridge exactly once, and return to the starting point.

Konigsberg bridges

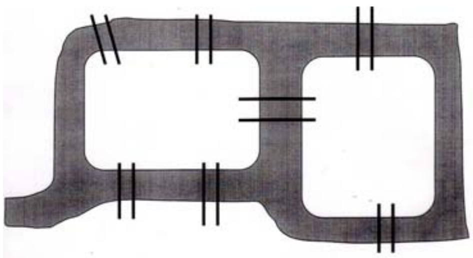
Is it possible to find a route that

- Starts and finishes at the same place?
- Crosses each bridge exactly once?



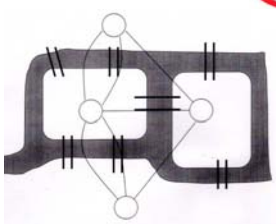
Konigsberg bridges

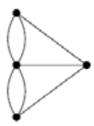
Simplify the problem



Konigsberg bridges

Model it as a graph where the edges represent the bridges and the vertices represent the islands.





Konigsberg bridges

In 1736 Leonard Euler proved that it was not possible because all the vertices of the graph are odd.

An **Eulerian** cycle travels along every edge in a network and returns to the starting point.

Then and now

In 1946 Konigsberg became part of the Soviet Union and it's name was changed to Kaliningrad.

Two of the seven original bridges were destroyed during World War II. Two others were later demolished and replaced by a motorway.

The three other bridges remain, although only two of them are from Euler's time (one was rebuilt in 1935).

Hence there are now only 5 bridges in Konigsberg (Kaliningrad).

Graphs

Many problems of practical interest can be represented by graphs.

- The link structure of a website
- Social relationships

The development of algorithms to handle graphs is therefore of major interest in computer science and electronics

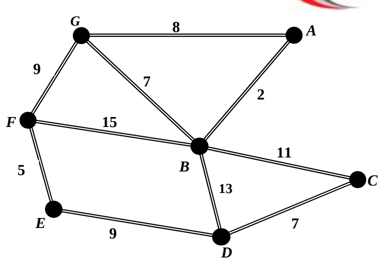
Networks

- When the edges of a graph have numbers (weights) it is called a network.
- Networks can be used to represent many different things; for example if the graph represents a road network, the weights could represent the length of each road.
- Network analysis can be used to find the shortest distance between two places or to model and analyse traffic flow

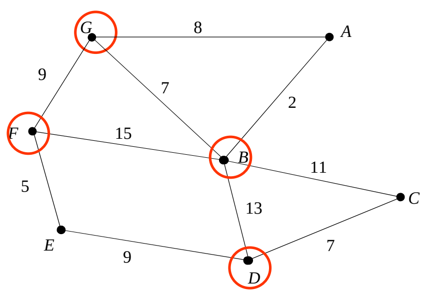
A problem

A postman starts his rounds at the depot. He needs to deliver letters along the all the streets and return to the depot at *D*. What is the shortest route he can take?

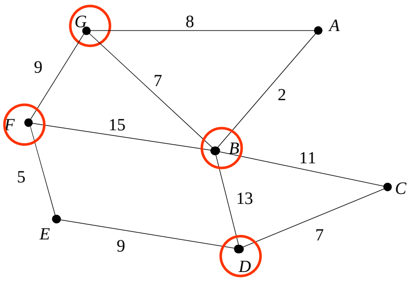
Distances are in 100 metres



Identify the odd nodes

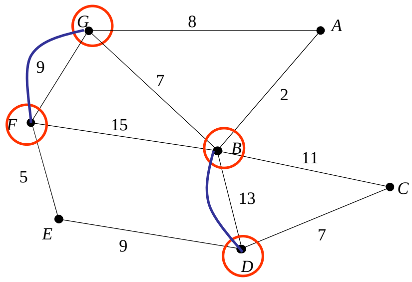


possible pairings are



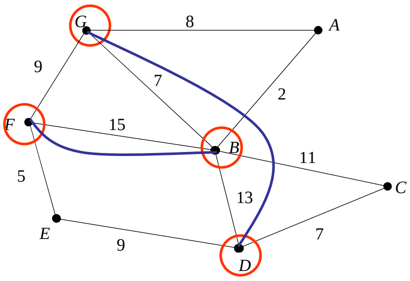
BD and FG
BF and DG
BG and DF

Consider possible pairings



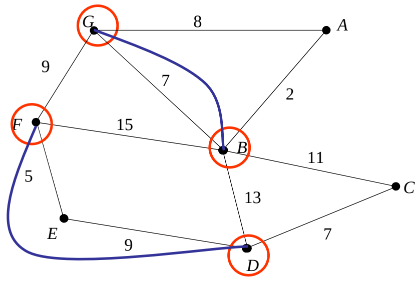
BD 13
FG 9
Extra distance
13 + 9 = 22

Consider possible pairings



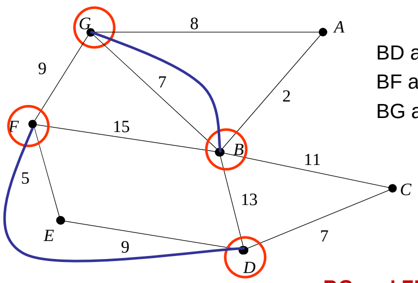
BF 15
DG 20 (via B)
Extra distance
15 + 20 = 35

Consider possible pairings



BG 7
DF 14 (via E)
Extra distance
7 + 14 = 21

Best solution

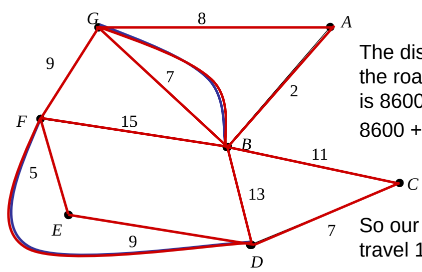


BD and FG	22
BF and DG	35
BG and DF	21

Don't forget DF goes via E

BG and FD is the best because it is the least extra distance

solution



The distance along all the roads in the Network is 8600m
8600 + 2100 = 10700m

So our postman must travel 10.7 Km

One possible route: DCB**DEF**GA**BGBFED**

Algorithms

- An algorithm is a set of instructions for solving a type of problem.
- Finding cycles that go along every edge at least once is called a **Route Inspection problem.**
- It is sometimes called the **Chinese Postman problem** after the Chinese mathematician, Mei Ko Kwan, who developed the algorithm in 1962

Decision Maths

What's it all about?

- Modelling with graphs and networks
- Using algorithms
- Scheduling
- Optimisation (linear programming)

What's it useful for?

- Widely used in the real world
- Operational Research
- Business, computing and electronic engineering