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Innovators in Mathematics Education

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The cubic and quartic formulae

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Scipione del Ferro (1465 – 1526) knew how to solve equations of the form $x^3 + mx = n$ and on his deathbed told Antonio Fior (1506 - ?) who challenged Niccolo "Tartaglia" Fontana (1499-1557) who could already solve equations of the form $x^3 + mx^2 = n$. He discovered the solution to $x^3 + mx = n$ on February 13th 1535 and so triumphed in the challenge. On March 25th 1539 he told, under oath of secrecy, Gerolamo Cardano (1501 – 1576) who, with his student Ludovico Ferrari (1522 – 1565) read del Ferro's work in 1543 and published Ars Magna in 1545.

Cardano's method for solving the general cubic.

1. Divide every term of your cubic $ax^3 + bx^2 + cx + d = 0$ by *a*, the coefficient of x^3.

2. Use the transformation, $x = y - \frac{b}{3a}$ to give a depressed cubic (i.e. one with no quadratic term): $y^3 + my = n$

3. Compare the identity $(t-u)^3 + 3tu(t-u) \equiv t^3 - u^3$ with your depressed cubic.

4. Solve the simultaneous equations $3tu = m$ and $t^3 - u^3 = n$ to find *t* and *u* in terms of *m* and *n*.

5. A root of the depressed cubic is $y = t - u$ and so a root of the original cubic is $x = y - \frac{b}{3a} = t - u - \frac{b}{3a}$

$$y = \sqrt[3]{\frac{n}{2} + \sqrt{\frac{n^2}{4} + \frac{m^3}{27}}} - \sqrt[3]{-\frac{n}{2} + \sqrt{\frac{n^2}{4} + \frac{m^3}{27}}}$$

Examples

1. $x^3 - 15x^2 + 81x - 175 = 0$
2. $x^3 - 15x = 4$
3. $x^3 + 9x = 26$
4. $x^3 - 3x^2 - 3x - 4 = 0$
5. $x^3 - 6x = 4$
6. $x^3 - 6x = 40$
7. $x^3 - 6x^2 + 11x - 6 = 0$
8. $x^3 - 6x^2 + 13x - 12 = 0$

Example. Find the roots of $x^3 - 15x^2 + 81x - 175 = 0$

First use the substitution $x = y + 5$ to get a depressed cubic:
$(y+5)^3 - 15(y+5)^2 + 81(y+5) - 175 = 0$ which simplifies to $y^3 + 6y = 20$.

Using the formula with $m = 6, n = 20$:

$$y = \sqrt[3]{\frac{20}{2} + \sqrt{\frac{20^2}{4} + \frac{6^3}{27}}} - \sqrt[3]{-\frac{20}{2} + \sqrt{\frac{20^2}{4} + \frac{6^3}{27}}}$$ which simplifies to

$$y = \sqrt[3]{10 + \sqrt{108}} - \sqrt[3]{-10 + \sqrt{108}} = \sqrt[3]{10 + 6\sqrt{3}} - \sqrt[3]{-10 + 6\sqrt{3}}$$

But $\left(1+\sqrt{3}\right)^3 = 10 + 6\sqrt{3}$ and $\left(-1+\sqrt{3}\right)^3 = -10 + 6\sqrt{3}$.
Therefore $y = \left(1+\sqrt{3}\right) - \left(-1+\sqrt{3}\right) = 2$

You can see that $y = 2$ is indeed a root of $y^3 + 6y = 20$.

It follows that $x = y + 5 = 2 + 5 = 7$ is a root of $x^3 - 15x^2 + 81x - 175 = 0$.

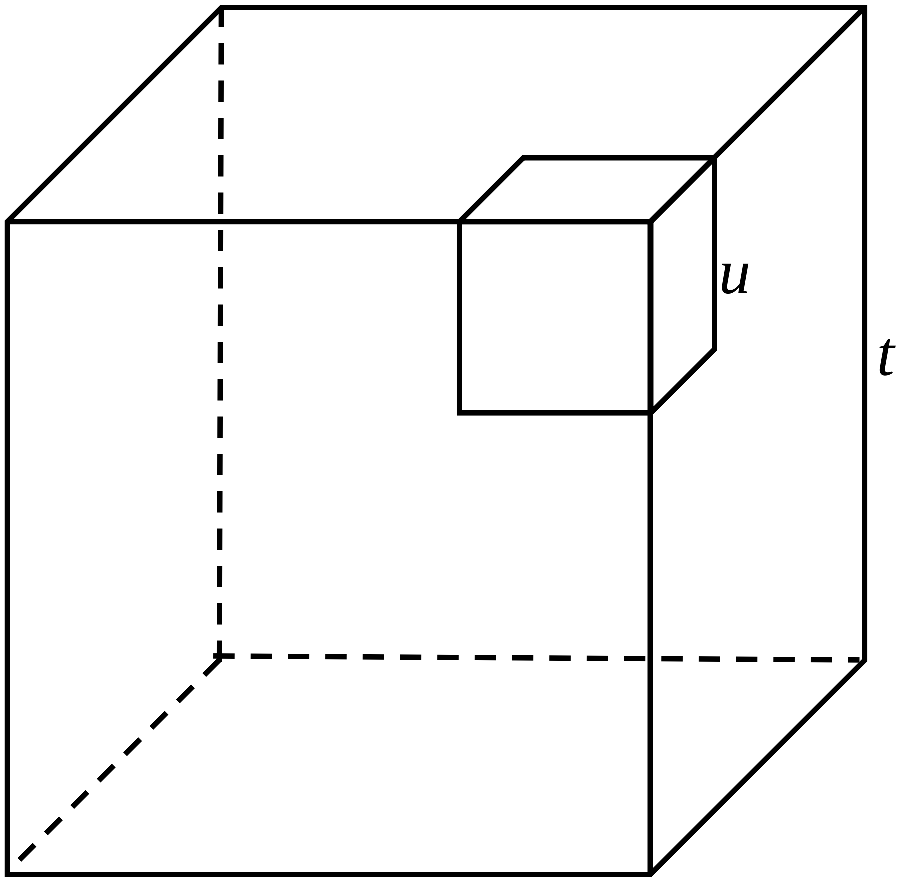
To find the other two roots we divide this polynomial by $(x-7)$:

$$x^3 - 15x^2 + 81x - 175 = (x-7)\left(x^2 - 8x + 25\right) = (x-7)\left((x-4)^2 + 9\right)$$

This shows that the roots of the equation $x^3 - 15x^2 + 81x - 175 = 0$ are $x = 7$ and the two imaginary roots of $x = 4 \pm 3i$.

You may prefer not to use the formula but go back to basics and compare the depressed cubic $y^3 + 6y = 20$ with the identity $(t-u)^3 + 3tu(t-u) \equiv t^3 - u^3$.
Then $3tu = 6$ and $t^3 - u^3 = 20$. Substituting $u = \frac{2}{t}$ in $t^3 - u^3 = 20$ leads to the quadratic in t^3: $\left(t^3\right)^2 - 20\left(t^3\right) - 8 = 0$ which, on completing the square, gives $\left(t^3 - 10\right)^2 = 108$ and so $t^3 = 10 \pm \sqrt{108} = 10 \pm 6\sqrt{3}$. Substituting this in $t^3 - u^3 = 20$ gives $u^3 = -10 \pm 6\sqrt{3}$.

Therefore $t - u = \sqrt[3]{10 + 6\sqrt{3}} - \sqrt[3]{-10 + 6\sqrt{3}}$ which simplifies to
$t - u = \left(1+\sqrt{3}\right) - \left(-1+\sqrt{3}\right) = 2$ and this is a root of the equation $y^3 + 6y = 20$.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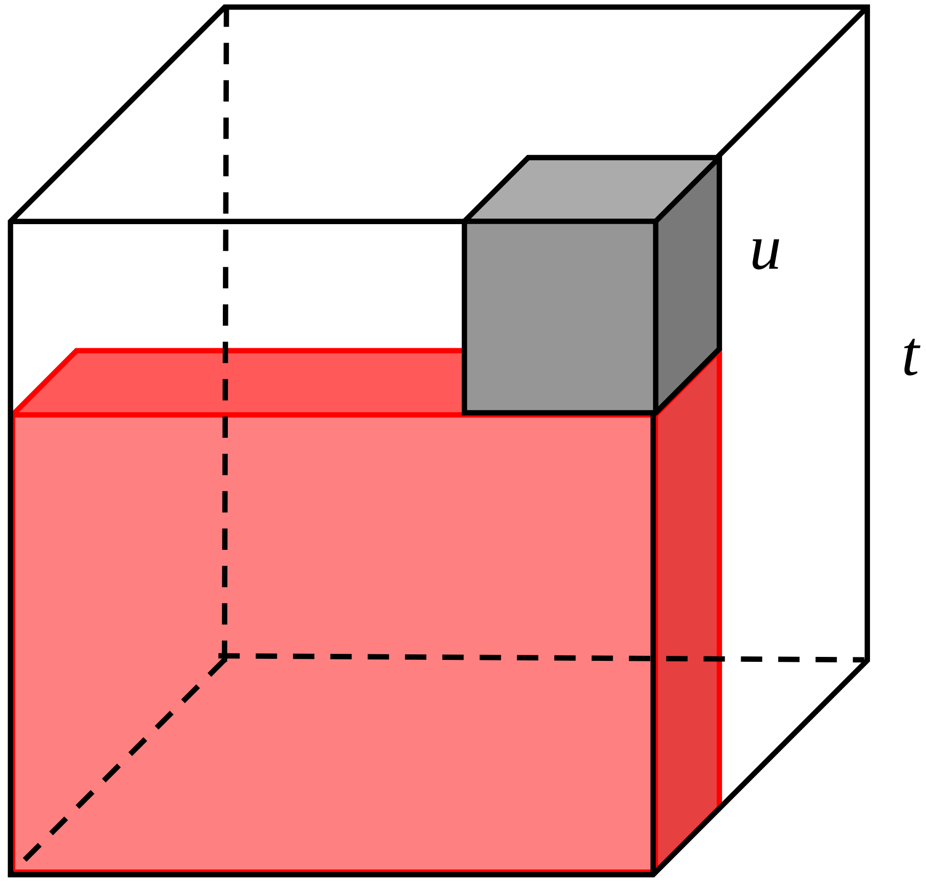
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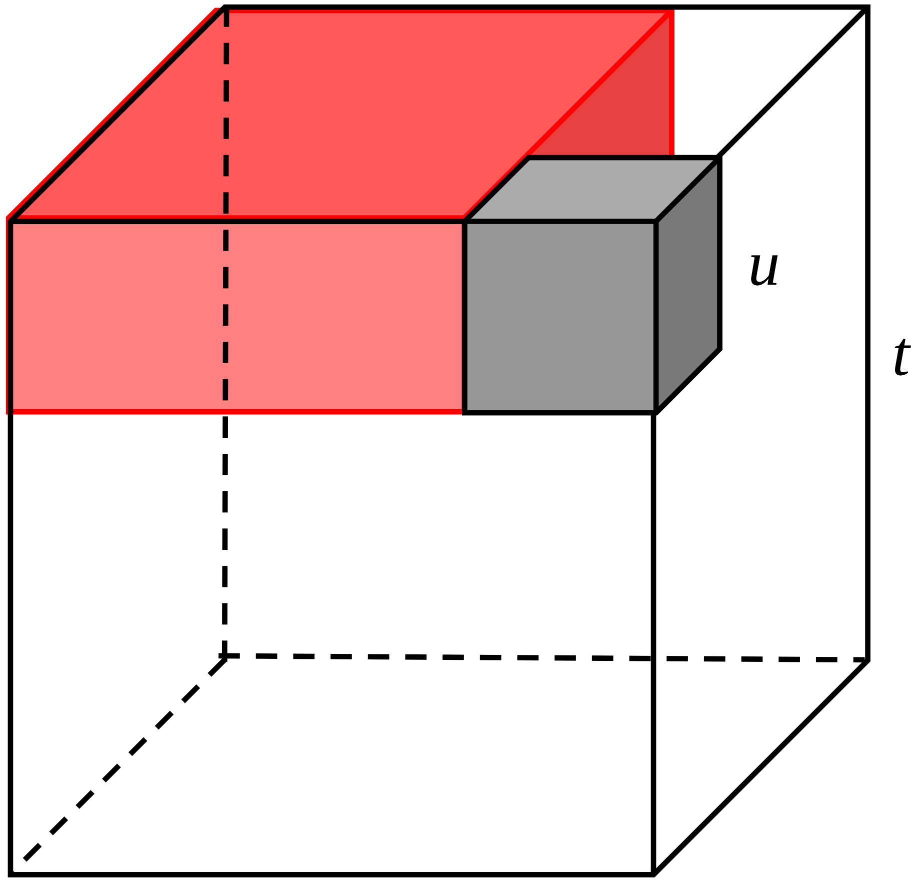
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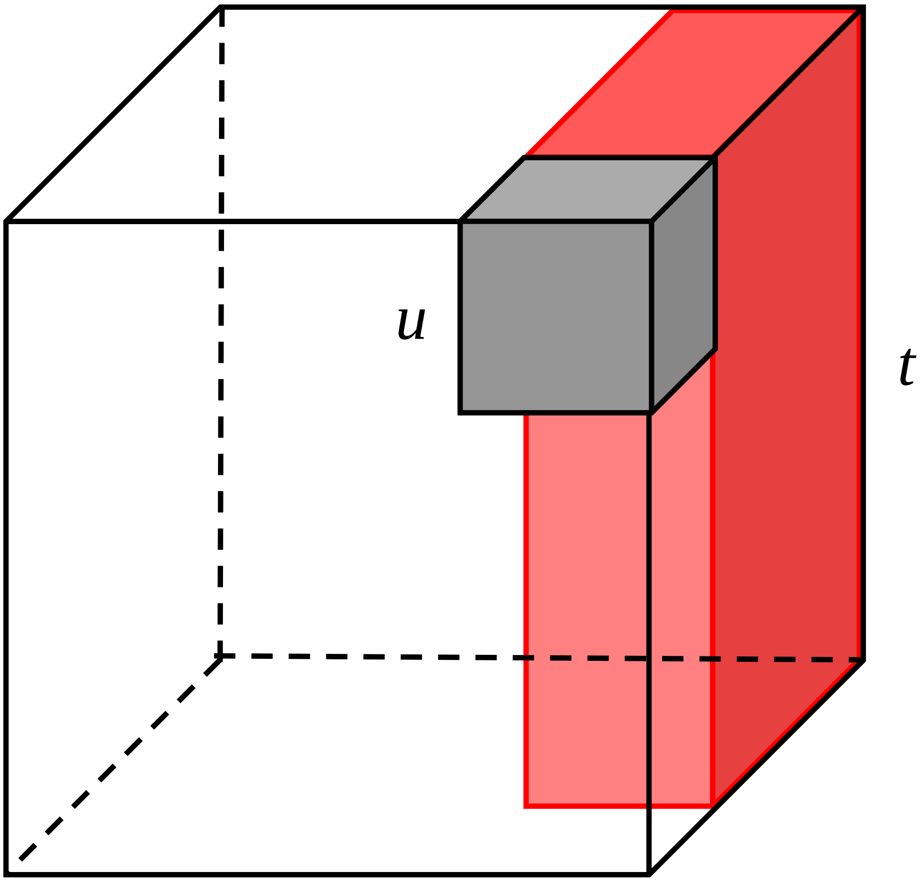
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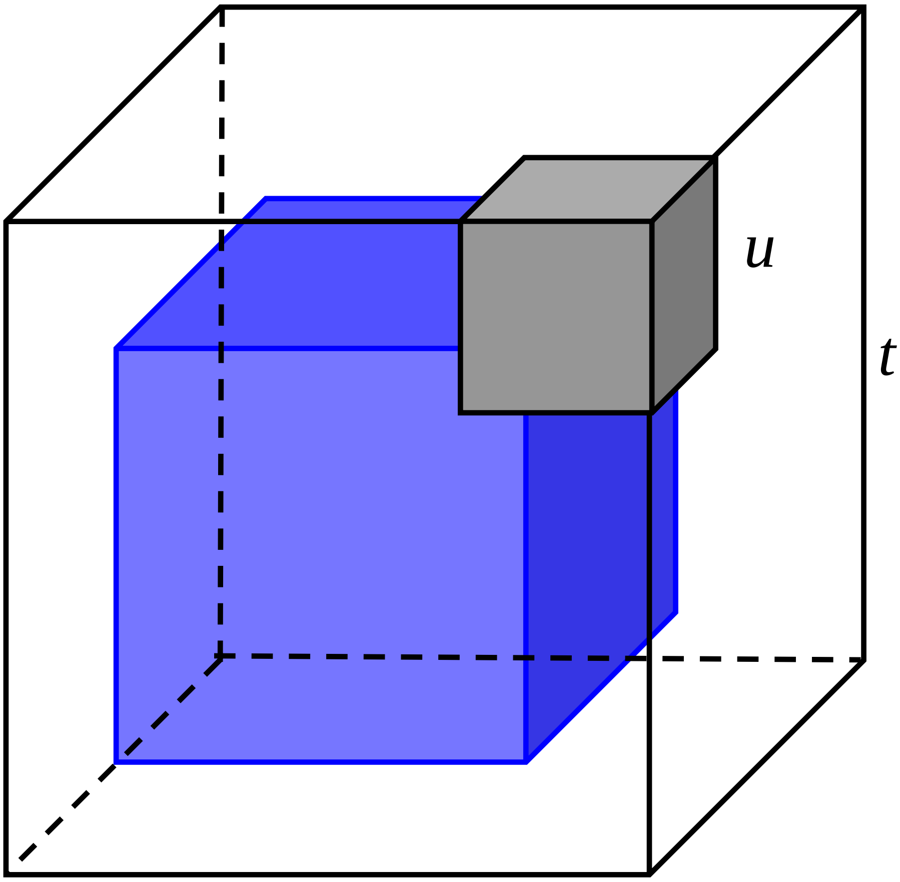
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Ferrari's method for solving the quartic.

1. Divide all terms of the quartic by the coefficient of x^4 to give
$$x^4+ax^3+bx^2+cx+d=0$$

2. Compare the coefficients of this quartic with those of the quartic
$$\left(x^2+\frac{a}{2}x+p\right)^2-\left(qx+r\right)^2=0$$

3. By eliminating *q* and *r* from the resulting simultaneous equations, form a cubic in *p*.

4. Use Cardano's method (or other) to find *p* and then, by substitution, *q* and *r*.

5. Consider the two quadratics $x^2+\frac{a}{2}x+p=\pm\left(qx+r\right)$

Example Find the roots of $x^4-4x^3-36x^2+16x+128=0$

Compare the quartic with $\left(x^2-2x+p\right)^2-\left(qx+r\right)^2=0$:

Coefficients of x^2: $2p+4-q^2=-36 \quad\Rightarrow\quad q^2=2p+40 \qquad (1)$

Coefficients of x: $-4p-2qr=16 \quad\Rightarrow\quad qr=-2p-8 \qquad (2)$

Constant term: $p^2-r^2=128 \quad\Rightarrow\quad r^2=p^2-128 \qquad (3)$

We now can now express q^2r^2 in two ways: $\left(-2p-8\right)^2=\left(2p+40\right)\left(p^2-128\right)$ which simplifies to $p^3+18p^2-144p-2592=0$.

Cardano's method could be used to find p but there's no need as the cubic factorises:
$\left(p+18\right)\left(p-12\right)\left(p+12\right)=0$

Using $p=12$ (see * below) in $(1),(2)$ and (3) gives $q=\pm8,\quad r=\pm4$ and $qr=-32$, the latter showing that q and r must have opposite signs.

Substituting these back into $\left(x^2-2x+p\right)^2-\left(qx+r\right)^2=0$ gives
$\left(x^2-2x+12\right)^2=\left(8x-4\right)^2$ and so $x^2-2x+12=\pm\left(8x-4\right)$.

This leads to the two quadratics:

$x^2-2x+12=+\left(8x-4\right) \quad\Rightarrow\quad x^2-10x+16=0 \quad\Rightarrow\quad \left(x-2\right)\left(x-8\right)=0$

and $x^2-2x+12=-\left(8x-4\right) \quad\Rightarrow\quad x^2+6x+8=0 \quad\Rightarrow\quad \left(x+2\right)\left(x+4\right)=0$

Therefore the four roots of the quartic are $x=2,-2,-4,8$

* Using either $p=-12$ or $p=18$ would give the same four roots in a different order.