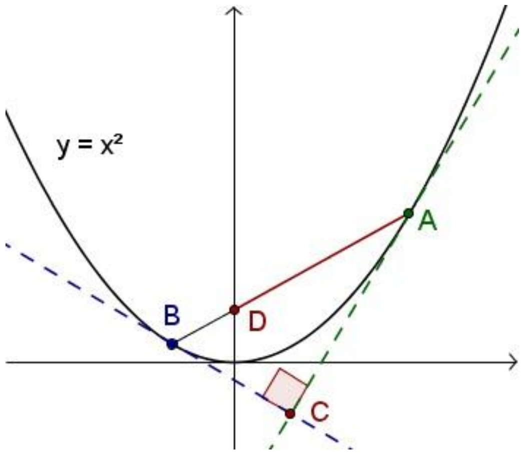
Maths Item of the Month – November 2011

Right on the curve

On $y = x^2$ plot a point one the curve, A, and draw the tangent to the curve at A. Plot the other tangent to the curve that is perpendicular to this tangent. Label the point at which this touches the curve B.



What are the coordinates of the point of intersection of the two tangents (shown as C on the diagram)?

What are the coordinates of the point of intersection of the line AB with the y-axis (labelled as D on the diagram).

Repeat this for different values of A.
What do you notice?

Would this work for other parabolas?

Solution

Taking the coordinates of A as (a, a^2) the gradient at a is $2a$.

The tangent through a is $y - a^2 = 2a(x - a)$ or $y = 2ax - a^2$

The tangent perpendicular to this has gradient $\frac{-1}{2a}$ and therefore touches the curve at $x = -\frac{1}{4a}$.

The coordinates of B are $\left(-\frac{1}{4a}, \frac{1}{16a^2}\right)$.

The tangent through B is $y - \frac{1}{16a^2} = \frac{-1}{2a}(x + \frac{1}{4a})$ or $y = \frac{-1}{2a}x - \frac{1}{16a^2}$

Solving to find the point of intersection of the tangents, C, gives: $x = \frac{4a^2 - 1}{8a}, y = -\frac{1}{4}$

i.e. the point of intersection lies on the line $y = -\frac{1}{4}$ which is the **directrix** of $y = x^2$.

The line joining A and B is $\frac{y - a^2}{x - a} = \frac{\frac{1}{16a^2} - a^2}{\frac{-1}{4a} - a}$.

This intercepts the y-axis at D: $\left(0, \frac{1}{4}\right)$, which is the **focus** of $y = x^2$.

For more information about the focus/directrix properties of parabolas see:
http://en.wikipedia.org/wiki/Parabola#Derivation_of_the_focus