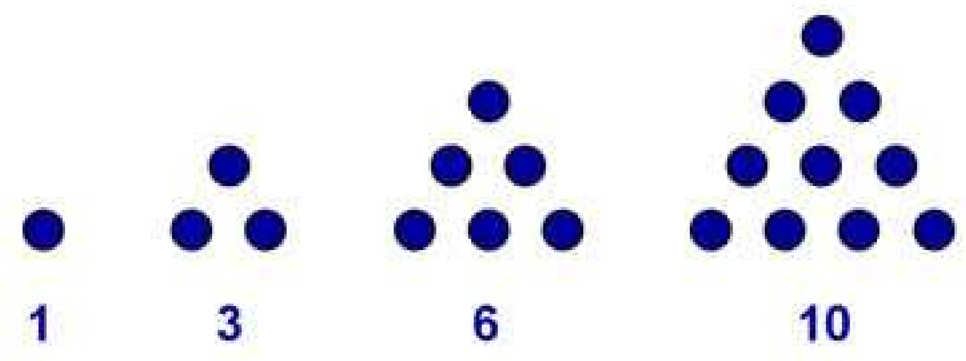
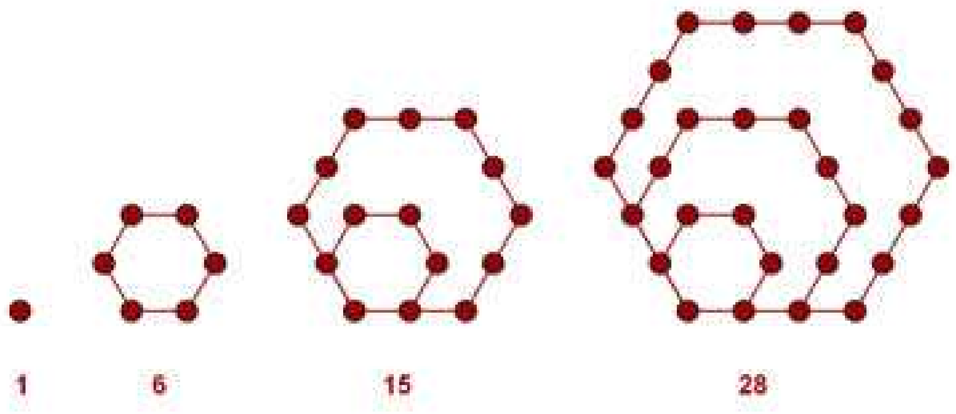
Not all triangles are perfect but ...

The first two perfect numbers are 6 and 28

A number is perfect if it is equal to the sum of its factors other than itself: e.g. 6 = 1 + 2 + 3 and 28 = 1+ 2 + 4 + 7 + 14.



The first four triangular numbers are 1, 3, 6, 10. Both 6 and 28 are triangular numbers. Are all perfect numbers triangular?



The first four hexagonal numbers are 1, 6, 15, 28. Both 6 and 28 are hexagonal numbers. Are all perfect numbers hexagonal?

Solution

All even perfect numbers are of the form $2^{p-1}(2^p-1)$, where p is prime and 2^p-1 is prime (a Mersenne prime).

The n^{th} triangular number, T_n is given by:

$$\begin{aligned} T_n &= 1+2+3+...+n \\ &= \frac{n(n+1)}{2} \end{aligned}$$

Rewriting the above form of an even perfect number gives:

$$2^{p-1}(2^p-1)=\frac{2^p(2^p-1)}{2}$$
$$=\frac{(2^p-1)((2^p-1)+1)}{2}$$

i.e. the $(2^p-1)^{\text{th}}$ triangular number.

The n^{th} hexagonal number, H_n is given by:

$$H_n=1+5+9+...+(4(n-1)+1)$$
$$=n(2n-1)$$

Rewriting the above form of an even perfect number gives:

$$2^{p-1}(2^p-1)=2^{p-1}(2\times 2^{p-1}-1)$$

i.e. the $(2^{p-1})^{\text{th}}$ hexagonal number.

These results hold for all even perfect numbers. To date no odd perfect numbers have been found and it is unknown whether any exist.