MEI Maths Item of the Month

August 2016

Square sum of squares

$1^2 + 2^2 = 5$, which is not a square.

$1^2 + 2^2 + 3^2 = 14$, which is not a square.

What is the smallest positive integer value of n, $n > 1$, such that $1^2 + 2^2 + ... + n^2$ is a square number?

Are there any larger possible values of n?

Solution

The smallest possible value of n is 24: $1^2 + 2^2 + ... + 24^2 = 4900$, $4900 = 70^2$

$\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$ therefore for $\sum_{r=1}^{n} r^2$ to be a square number we need either two of n, $n+1$ and $2n+1$ to be square and the other to be a square multiplied by 6 or for one them to be square, one to be a square multiplied by 2 and one to be a square multiplied by 3.

n	$n+1$	$2n+1$	
2	3	5	5 is not 1,2,3 or 6 × a square
3	4	7	7 is not 1,2,3 or 6 × a square
4	5	9	5 is not 1,2,3 or 6 × a square
5	6	11	5 is not 1,2,3 or 6 × a square
6	7	13	7 is not 1,2,3 or 6 × a square
7	8	15	7 is not 1,2,3 or 6 × a square
8	9	17	17 is not 1,2,3 or 6 × a square
9	10	19	19 is not 1,2,3 or 6 × a square
10	11	21	11 is not 1,2,3 or 6 × a square
11	12	23	11 is not 1,2,3 or 6 × a square
12	13	25	13 is not 1,2,3 or 6 × a square
13	14	27	13 is not 1,2,3 or 6 × a square
14	15	29	29 is not 1,2,3 or 6 × a square
15	16	31	31 is not 1,2,3 or 6 × a square
16	17	33	17 is not 1,2,3 or 6 × a square
17	18	35	17 is not 1,2,3 or 6 × a square
18	19	37	19 is not 1,2,3 or 6 × a square
19	20	39	19 is not 1,2,3 or 6 × a square
20	21	41	41 is not 1,2,3 or 6 × a square
21	22	43	43 is not 1,2,3 or 6 × a square
22	23	45	23 is not 1,2,3 or 6 × a square
23	24	47	23 is not 1,2,3 or 6 × a square
24	25	49	$24=6\times2^2$, $25=5^2$, $49=7^2$

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There are no larger values: $\sum_{r=1}^{n} r^2 = N^2$ has a unique solution in integers for $n > 1$.

n, $(n+1)$ and $(2n+1)$ do not share any common factors therefore for $\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$ to be square the terms in the numerator must be square numbers multiplied by coefficients with product 6.

n	$n+1$	$2n+1$	*Reason why not possible*
a^2	b^2	$6c^2$	$2n+1$ is an odd number but $6c^2$ is even.
a^2	$2b^2$	$3c^2$	$n+1=2b^2 \Rightarrow 2n+1=4b^2-1$ which can't be square since congruent to 3(mod4)
a^2	$3b^2$	$2c^2$	$2n+1$ is an odd number but $2c^2$ is even.
a^2	$6b^2$	c^2	$n+1=6b^2 \Rightarrow 2n+1=12b^2-1$ which can't be square since congruent to 11(mod12)
$2a^2$	b^2	$3c^2$	$n=2a^2 \Rightarrow 2n+1=4a^2+1$ but this is not divisible by 3: $4(3k\pm1)^2+1 \equiv 2(\bmod 3)$ and $4(3k)^2+1 \equiv 1(\bmod 3)$
$2a^2$	$3b^2$	c^2	$n+1=3b^2 \Rightarrow 2n+1=6b^2-1$ which can't be square since congruent to 5(mod6)
$3a^2$	b^2	$2c^2$	$2n+1$ is an odd number but $2c^2$ is even.
$3a^2$	$2b^2$	c^2	$n+1=2b^2 \Rightarrow 2n+1=4b^2-1$ which can't be square since congruent to 3(mod4)
$6a^2$	b^2	c^2	

This leaves only $n=6a^2, n+1=b^2, 2n+1=c^2$.

Substituting in $n=6a^2$ gives $c^2-b^2=(c-b)(c+b)=6a^2$.

Note that $a=1$ does not lead to a solution (since $6\times1^2+1$ is not square) and that a cannot be a factor of both $c-b$ and $c+b$.

So the factors could be:

$c-b$	$c+b$	*Reason why not possible*
1	$6a^2$	$2c=1+6a^2$ would not give an integer for c
2	$3a^2$	
3	$2a^2$	$2c=3+2a^2$ would not give an integer for c
6	a^2	$c^2=(b+6)^2=b^2+12b+36$. This leads to $n=12b+36$ and so $b^2-1=12b+36$ which does not have integer solutions.
a^2	6	$c>b>1$. Only possibility is $c=4, b=2$ but doesn't lead to a solution

This leaves only $c-b=2, c+b=3a^2$. Substitute $c=b+2$ in $n+1=b^2, 2n+1=c^2$ leads to $b^2-4b-5=0$. Then $b=5, c=7, a=2$ giving $n=24, n+1=25, 2n+1=49$.