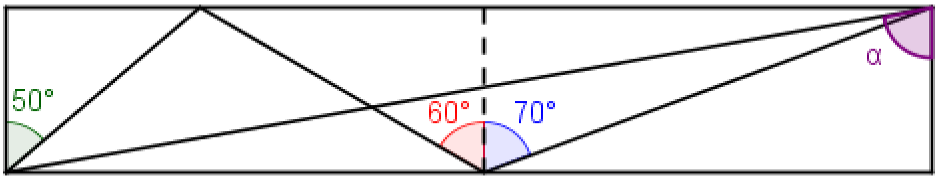
MEI Maths Item of the Month

February 2016

50, 60, 70, ... ?

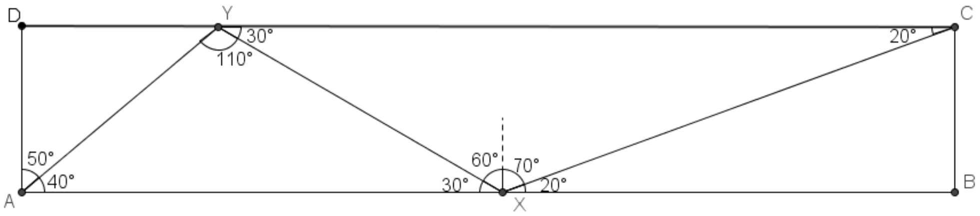
Find the size of the angle α.



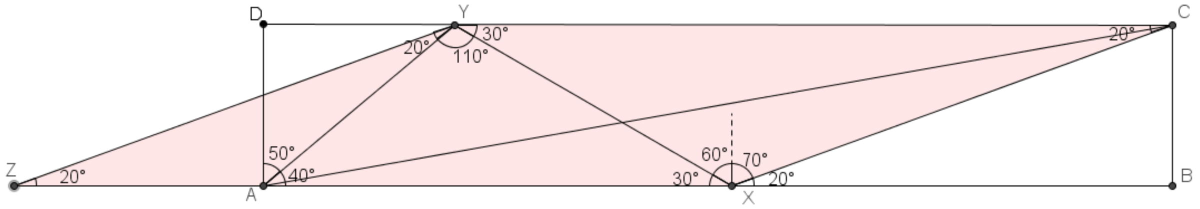
Solution *(submitted by Howard Fay)*

α = 80°.

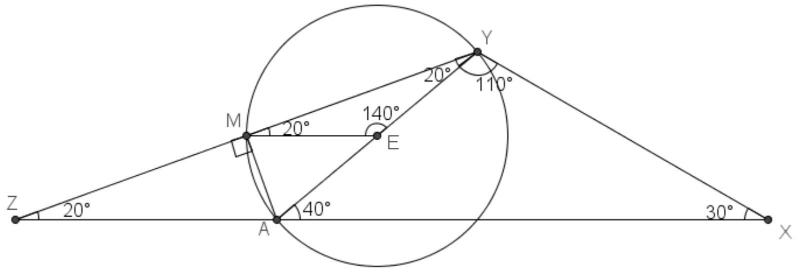
In rectangle ABCD, BC=1 unit and DC = tan50° + tan60° + tan70°.



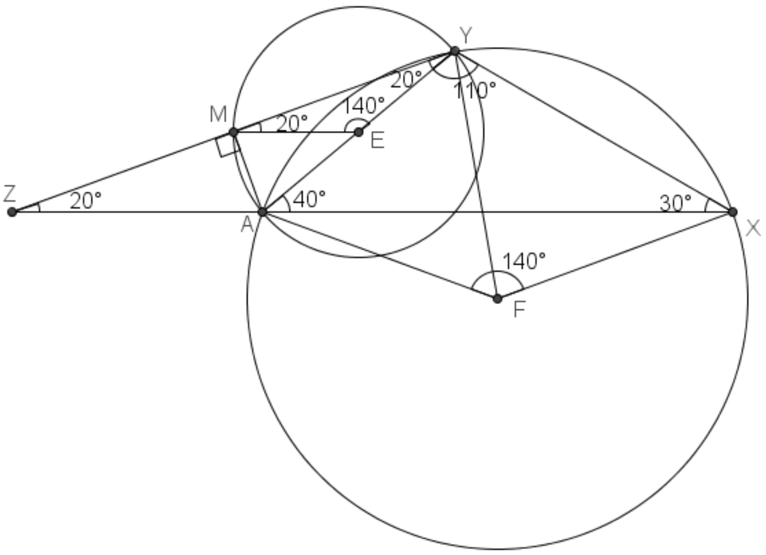
XCYZ is a parallelogram and triangle ZAY must be isosceles.
To prove tan50° + tan60° + tan70° = tan80° we need to prove that angle ACY = 10°.
Since angle AXC = 160°, this is equivalent to proving that AX = XC.
This is equivalent to proving that AX = ZY.



M is the midpoint of ZY and E is the midpoint of AY.
The circle with diameter AY passes through M.



The circle passing through A, Y and X has centre F.
Angle AFY = 60° (double angle AXY) and so AY = radius AF.
Obtuse angle AFX = 140° (reflex angle AFX = double angle AYX).



Isosceles triangle AFX is similar to triangle MEY.

Ratio of sides: $\frac{ME}{AF} = \frac{\frac{1}{2}AY}{AY} = \frac{1}{2}$.

Therefore $\frac{1}{2} = \frac{MY}{AX} = \frac{\frac{1}{2}ZY}{AX}$ and so AX = ZY.