MEI Maths Item of the Month

June 2016

MEI Conference taster – Odd and distinct partitions

O(n) is the number of ways of writing n as the sum of odd positive integers.
e.g. O(6) = 4: {5+1, 3+3, 3+1+1, 1+1+1+1+1+1}

D(n) is the number of ways of writing n as the sum of distinct positive integers.
e.g. D(6) = 4: {6, 5+1, 4+2, 3+2+1}

Does O(n) = D(n) for all natural numbers?

This problem is taken from the 2015 MEI Conference session *Desert Island Mathematics*.

Solution

This is Euler's proof of this result.

First, consider the function $\mathrm{P}(x)=(1+x)(1+x^2)(1+x^3)(1+x^4)(1+x^5)\ldots$

In the expansion, x^6 arises in exactly 4 ways; this 4 corresponding to $\mathrm{D}(6)$:

$$x^1.x^2.x^3.1.1.1.1\ldots \quad + \quad x^1.1.1.1.x^5.1.1\ldots \quad + \quad 1.x^2.1.x^4.1.1.1.\ldots \quad + \quad 1.1.1.1.1.x^6.1.\ldots$$

It can be seen that $\mathrm{P}(x)=1+\mathrm{D}(1)x+\mathrm{D}(2)x^2+\mathrm{D}(3)x^3+\mathrm{D}(4)x^4+\ldots$

Now consider the function $\mathrm{Q}(x)=\dfrac{1}{1-x}.\dfrac{1}{1-x^3}.\dfrac{1}{1-x^5}.\dfrac{1}{1-x^7}.\dfrac{1}{1-x^9}\ldots$

Thinking about infinite geometric series, or the binomial theorem, this is the same as

$$\mathrm{Q}(x)=(1+x^1+x^{1+1}+x^{1+1+1}+\ldots)(1+x^3+x^{3+3}+x^{3+3+3}+\ldots)(1+x^5+x^{5+5}+x^{5+5+5}+\ldots)\ldots$$

In the expansion, x^6 arises in exactly 4 ways; this 4 corresponding to $\mathrm{O}(6)$:

$$x^{1+1+1+1+1+1}.1.1.1\ldots \quad + \quad x^{1+1+1}.x^3.1.1\ldots \quad + \quad x^1.1.x^5.1.\ldots \quad + \quad 1.x^{3+3}.1.1\ldots$$

It can be seen that $\mathrm{Q}(x)=1+\mathrm{O}(1)x+\mathrm{O}(2)x^2+\mathrm{O}(3)x^3+\mathrm{O}(4)x^4+\ldots$

From the outset, Euler realised that $\mathrm{P}(x)\equiv\mathrm{Q}(x)$:

$$\begin{aligned}\mathrm{P}(x)&=(1+x)(1+x^2)(1+x^3)(1+x^4)(1+x^5)\ldots\\&=(1+x)\times1\times(1+x^2)\times1\times(1+x^3)\times1\times(1+x^4)\times1\times(1+x^5)\ldots\\&=\cancel{(1+x)}\frac{\cancel{(1-x)}}{(1-x)}\cancel{(1+x^2)}\frac{\cancel{(1-x^2)}}{\cancel{(1-x^2)}}(1+x^3)\frac{(1-x^3)}{(1-x^3)}(1+x^4)\frac{(1-x^4)}{\cancel{(1-x^4)}}\ldots=\mathrm{Q}(x)\end{aligned}$$

Therefore $\mathrm{D}(n)=\mathrm{O}(n)$ for all values of n.