Maths Item of the Month
August 2011

A fraction of the Pythagorean triples

$\frac{1}{3}+\frac{1}{5}=\frac{8}{15}$ and $(8,15,17)$ is a Pythagorean triple.

Add the reciprocals of any two consecutive odd numbers. Will the resulting fraction, $\frac{x}{y}$, always generates an integer Pythagorean triple, (x,y,z)?

Solution

By writing the consecutive odd numbers as $2n-1$ and $2n+1$ the sum of the fractions is:

$$\begin{aligned}\frac{1}{2n-1}+\frac{1}{2n+1}&=\frac{2n-1+2n+1}{(2n+1)(2n-1)}\\&=\frac{4n}{4n^2-1}\end{aligned}$$

The conjecture is that this generates a fraction $\frac{a}{b}$ such that $a^2+b^2=c^2$ where a, b and c are all integers.

Squaring and adding the terms for a and b gives:

$$\begin{aligned}(4n)^2+(4n^2-1)^2&=16n^2+16n^4-8n^2+1\\&=16n^4+8n^2+1\\&=(4n^2+1)^2\end{aligned}$$

$(4n+1)$ will be an integer therefore $(4n,4n^2-1,4n^2+1)$ is a Pythagorean triple.