March 2010 Difference of two square roots

$$\left(\sqrt{2}-1\right)^2 = \sqrt{9}-\sqrt{8}$$
$$\left(\sqrt{2}-1\right)^3 = \sqrt{50}-\sqrt{49}$$

Is every positive integer power of $\left(\sqrt{2}-1\right)$ the difference between the square roots of consecutive integers?

Solution

This can be proved by induction, taking even powers and odd powers separately.

Even powers.

$\left(\sqrt{2}-1\right)^2=\sqrt{9}-\sqrt{8}$ [or the starting point could be $\left(\sqrt{2}-1\right)^0=\sqrt{1}-\sqrt{0}$]

Assume $\left(\sqrt{2}-1\right)^{2k}=A-B\sqrt{2}=\sqrt{A^2}-\sqrt{2B^2}$ where $A^2-2B^2=1$

Then
$$\begin{aligned}\left(\sqrt{2}-1\right)^{2k+2}&=\left(\sqrt{2}-1\right)^2\left(\sqrt{2}-1\right)^{2k}\\&=\left(3-2\sqrt{2}\right)\left(A-B\sqrt{2}\right)\\&=\left(3A+4B\right)-\sqrt{2}\left(2A+3B\right)\\&=\sqrt{\left(3A+4B\right)^2}-\sqrt{2\left(2A+3B\right)^2}\end{aligned}$$

Now $\left(3A+4B\right)^2-2\left(2A+3B\right)^2=A^2-2B^2=1$

So $\left(\sqrt{2}-1\right)^{2k}$ takes the desired form.

Odd powers.

$\left(\sqrt{2}-1\right)^1=\sqrt{2}-\sqrt{1}$

Assume $\left(\sqrt{2}-1\right)^{2k-1}=C\sqrt{2}-D=\sqrt{2C^2}-\sqrt{D^2}$ where $2C^2-D^2=1$

Then
$$\begin{aligned}\left(\sqrt{2}-1\right)^{2k+1}&=\left(\sqrt{2}-1\right)^2\left(\sqrt{2}-1\right)^{2k-1}\\&=\left(3-2\sqrt{2}\right)\left(C\sqrt{2}-D\right)\\&=\sqrt{2}\left(3C+2D\right)-\left(4C+3D\right)\end{aligned}$$

Now $2\left(3C+2D\right)^2-\left(4C+3D\right)^2=2C^2-D^2=1$

So $\left(\sqrt{2}-1\right)^{2k+1}$ also takes the desired form.