

**Monday 23 June 2014 – Morning**

**A2 GCE MATHEMATICS (MEI)**

**4756/01** Further Methods for Advanced Mathematics (FP2)

**QUESTION PAPER**

Candidates answer on the Printed Answer Book.

**OCR supplied materials:**

- Printed Answer Book 4756/01
- MEI Examination Formulae and Tables (MF2)

**Other materials required:**

- Scientific or graphical calculator

**Duration:** 1 hour 30 minutes



**INSTRUCTIONS TO CANDIDATES**

These instructions are the same on the Printed Answer Book and the Question Paper.

- The Question Paper will be found inside the Printed Answer Book.
- Write your name, centre number and candidate number in the spaces provided on the Printed Answer Book. Please write clearly and in capital letters.
- **Write your answer to each question in the space provided in the Printed Answer Book.** Additional paper may be used if necessary but you must clearly show your candidate number, centre number and question number(s).
- Use black ink. HB pencil may be used for graphs and diagrams only.
- Read each question carefully. Make sure you know what you have to do before starting your answer.
- Answer **all** the questions.
- Do **not** write in the bar codes.
- You are permitted to use a scientific or graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.

**INFORMATION FOR CANDIDATES**

This information is the same on the Printed Answer Book and the Question Paper.

- The number of marks is given in brackets [ ] at the end of each question or part question on the Question Paper.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.
- The total number of marks for this paper is **72**.
- The Printed Answer Book consists of **16** pages. The Question Paper consists of **4** pages. Any blank pages are indicated.

**INSTRUCTION TO EXAMS OFFICER/INVIGILATOR**

- Do not send this Question Paper for marking; it should be retained in the centre or recycled. Please contact OCR Copyright should you wish to re-use this document.

## Section A (54 marks)

- 1 (a) Given that  $f(x) = \arccos x$ ,
- (i) sketch the graph of  $y = f(x)$ , [2]
- (ii) show that  $f'(x) = -\frac{1}{\sqrt{1-x^2}}$ , [3]
- (iii) obtain the Maclaurin series for  $f(x)$  as far as the term in  $x^3$ . [7]

(b) A curve has polar equation  $r = \theta + \sin \theta$ ,  $\theta \geq 0$ .

- (i) By considering  $\frac{dr}{d\theta}$  show that  $r$  increases as  $\theta$  increases.

Sketch the curve for  $0 \leq \theta \leq 4\pi$ . [4]

- (ii) You are given that  $\sin \theta \approx \theta$  for small  $\theta$ . Find in terms of  $\alpha$  the approximate area bounded by the curve and the lines  $\theta = 0$  and  $\theta = \alpha$ , where  $\alpha$  is small. [3]

- 2 (a) The infinite series  $C$  and  $S$  are defined as follows.

$$C = a \cos \theta + a^2 \cos 2\theta + a^3 \cos 3\theta + \dots,$$

$$S = a \sin \theta + a^2 \sin 2\theta + a^3 \sin 3\theta + \dots,$$

where  $a$  is a real number and  $|a| < 1$ .

By considering  $C + jS$ , show that

$$S = \frac{a \sin \theta}{1 - 2a \cos \theta + a^2}.$$

Find a corresponding expression for  $C$ . [8]

- (b) P is one vertex of a regular hexagon in an Argand diagram. The centre of the hexagon is at the origin. P corresponds to the complex number  $\sqrt{3} + j$ .

- (i) Find, in the form  $x + jy$ , the complex numbers corresponding to the other vertices of the hexagon. [5]

- (ii) The six complex numbers corresponding to the vertices of the hexagon are squared to form the vertices of a new figure. Find, in the form  $x + jy$ , the vertices of the new figure. Find the area of the new figure. [4]

- 3 (a) (i) Find the eigenvalues and corresponding eigenvectors for the matrix  $\mathbf{A}$ , where

$$\mathbf{A} = \begin{pmatrix} 6 & -3 \\ 4 & -1 \end{pmatrix}. \quad [5]$$

- (ii) Write down a matrix  $\mathbf{P}$  and a diagonal matrix  $\mathbf{D}$  such that  $\mathbf{A} = \mathbf{PDP}^{-1}$ . [2]

- (b) (i) The  $3 \times 3$  matrix  $\mathbf{B}$  has characteristic equation

$$\lambda^3 - 4\lambda^2 - 3\lambda - 10 = 0.$$

Show that 5 is an eigenvalue of  $\mathbf{B}$ . Show that  $\mathbf{B}$  has no other real eigenvalues. [4]

- (ii) An eigenvector corresponding to the eigenvalue 5 is  $\begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix}$ .

Evaluate  $\mathbf{B} \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix}$  and  $\mathbf{B}^2 \begin{pmatrix} 4 \\ -2 \\ -8 \end{pmatrix}$ .

Solve the equation  $\mathbf{B} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -20 \\ 10 \\ 40 \end{pmatrix}$  for  $x, y, z$ . [4]

- (iii) Show that  $\mathbf{B}^4 = 19\mathbf{B}^2 + 22\mathbf{B} + 40\mathbf{I}$ . [3]

### Section B (18 marks)

- 4 (i) Given that  $\sinh y = x$ , show that

$$y = \ln(x + \sqrt{1+x^2}). \quad (*)$$

Differentiate (\*) to show that

$$\frac{dy}{dx} = \frac{1}{\sqrt{1+x^2}}. \quad [8]$$

- (ii) Find  $\int \frac{1}{\sqrt{25+4x^2}} dx$ , expressing your answer in logarithmic form. [3]

- (iii) Use integration by substitution with  $2x = 5 \sinh u$  to show that

$$\int \sqrt{25+4x^2} dx = \frac{25}{4} \left( \ln \left( \frac{2x}{5} + \sqrt{1 + \frac{4x^2}{25}} \right) + \frac{2x}{5} \sqrt{1 + \frac{4x^2}{25}} \right) + c,$$

where  $c$  is an arbitrary constant. [7]

**END OF QUESTION PAPER**



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**4756/01** Further Methods for Advanced Mathematics (FP2)

**PRINTED ANSWER BOOK**

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- Scientific or graphical calculator

**Duration:** 1 hour 30 minutes



Candidate forename		Candidate surname	
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Centre number						Candidate number				
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**Section A (54 marks)**

<b>1 (a) (i)</b>	
<b>1 (a) (ii)</b>	

<b>1 (a) (iii)</b>	

<b>1(b)(i)</b>	

<b>1 (b) (ii)</b>	

<b>2 (a)</b>	

(answer space continued on next page)



<b>2(a)</b>	<b>(continued)</b>

<b>2(b)(i)</b>	

<b>2(b)(ii)</b>	

<b>3(a)(i)</b>	

<b>3(a)(ii)</b>	
<b>3(b)(i)</b>	



<b>3 (b) (iii)</b>	





<b>4 (i)</b> (continued)	
<b>4 (ii)</b>	

**4 (iii)****(answer space continued on next page)**

<b>4 (iii) (continued)</b>	

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**GCE**

**Mathematics (MEI)**

Unit **4756**: Further Methods for Advanced Mathematics

Advanced GCE

**Mark Scheme for June 2014**

## 1. Annotations and abbreviations

Annotation in scoris	Meaning
<b>BP</b>	Blank Page – this annotation <b>must</b> be used on all blank pages within an answer booklet (structured or unstructured) and on each page of an additional object where there is no candidate response.
✓ and ✖	
BOD	Benefit of doubt
FT	Follow through
ISW	Ignore subsequent working
M0, M1	Method mark awarded 0, 1
A0, A1	Accuracy mark awarded 0, 1
B0, B1	Independent mark awarded 0, 1
SC	Special case
^	Omission sign
MR	Misread
Highlighting	
Other abbreviations in mark scheme	Meaning
E1	Mark for explaining
U1	Mark for correct units
G1	Mark for a correct feature on a graph
M1 dep*	Method mark dependent on a previous mark, indicated by *
cao	Correct answer only
oe	Or equivalent
rot	Rounded or truncated
soi	Seen or implied
www	Without wrong working

**2. Subject-specific Marking Instructions for GCE Mathematics (MEI) Pure strand**

- a Annotations should be used whenever appropriate during your marking.

**The A, M and B annotations must be used on your standardisation scripts for responses that are not awarded either 0 or full marks.** It is vital that you annotate standardisation scripts fully to show how the marks have been awarded.

For subsequent marking you must make it clear how you have arrived at the mark you have awarded.

- b An element of professional judgement is required in the marking of any written paper. Remember that the mark scheme is designed to assist in marking incorrect solutions. Correct *solutions* leading to correct answers are awarded full marks but work must not be judged on the answer alone, and answers that are given in the question, especially, must be validly obtained; key steps in the working must always be looked at and anything unfamiliar must be investigated thoroughly.

Correct but unfamiliar or unexpected methods are often signalled by a correct result following an *apparently* incorrect method. Such work must be carefully assessed. When a candidate adopts a method which does not correspond to the mark scheme, award marks according to the spirit of the basic scheme; if you are in any doubt whatsoever (especially if several marks or candidates are involved) you should contact your Team Leader.

- c The following types of marks are available.

**M**

A suitable method has been selected and *applied* in a manner which shows that the method is essentially understood. Method marks are not usually lost for numerical errors, algebraic slips or errors in units. However, it is not usually sufficient for a candidate just to indicate an intention of using some method or just to quote a formula; the formula or idea must be applied to the specific problem in hand, eg by substituting the relevant quantities into the formula. In some cases the nature of the errors allowed for the award of an M mark may be specified.

**A**

Accuracy mark, awarded for a correct answer or intermediate step correctly obtained. Accuracy marks cannot be given unless the associated Method mark is earned (or implied). Therefore M0 A1 cannot ever be awarded.

**B**

Mark for a correct result or statement independent of Method marks.

**E**

A given result is to be established or a result has to be explained. This usually requires more working or explanation than the establishment of an unknown result.

Unless otherwise indicated, marks once gained cannot subsequently be lost, eg wrong working following a correct form of answer is ignored. Sometimes this is reinforced in the mark scheme by the abbreviation isw. However, this would not apply to a case where a candidate passes through the correct answer as part of a wrong argument.

- d When a part of a question has two or more 'method' steps, the M marks are in principle independent unless the scheme specifically says otherwise; and similarly where there are several B marks allocated. (The notation 'dep \*' is used to indicate that a particular mark is dependent on an earlier, asterisked, mark in the scheme.) Of course, in practice it may happen that when a candidate has once gone wrong in a part of a question, the work from there on is worthless so that no more marks can sensibly be given. On the other hand, when two or more steps are successfully run together by the candidate, the earlier marks are implied and full credit must be given.
- e The abbreviation ft implies that the A or B mark indicated is allowed for work correctly following on from previously incorrect results. Otherwise, A and B marks are given for correct work only — differences in notation are of course permitted. A (accuracy) marks are not given for answers obtained from incorrect working. When A or B marks are awarded for work at an intermediate stage of a solution, there may be various alternatives that are equally acceptable. In such cases, exactly what is acceptable will be detailed in the mark scheme rationale. If this is not the case please consult your Team Leader.

Sometimes the answer to one part of a question is used in a later part of the same question. In this case, A marks will often be 'follow through'. In such cases you must ensure that you refer back to the answer of the previous part question even if this is not shown within the image zone. You may find it easier to mark follow through questions candidate-by-candidate rather than question-by-question.

- f Wrong or missing units in an answer should not lead to the loss of a mark unless the scheme specifically indicates otherwise. Candidates are expected to give numerical answers to an appropriate degree of accuracy, with 3 significant figures often being the norm. Small variations in the degree of accuracy to which an answer is given (e.g. 2 or 4 significant figures where 3 is expected) should not normally be penalised, while answers which are grossly over- or under-specified should normally result in the loss of a mark. The situation regarding any particular cases where the accuracy of the answer may be a marking issue should be detailed in the mark scheme rationale. If in doubt, contact your Team Leader.
- g Rules for replaced work
- If a candidate attempts a question more than once, and indicates which attempt he/she wishes to be marked, then examiners should do as the candidate requests.

If there are two or more attempts at a question which have not been crossed out, examiners should mark what appears to be the last (complete) attempt and ignore the others.

NB Follow these maths-specific instructions rather than those in the assessor handbook.

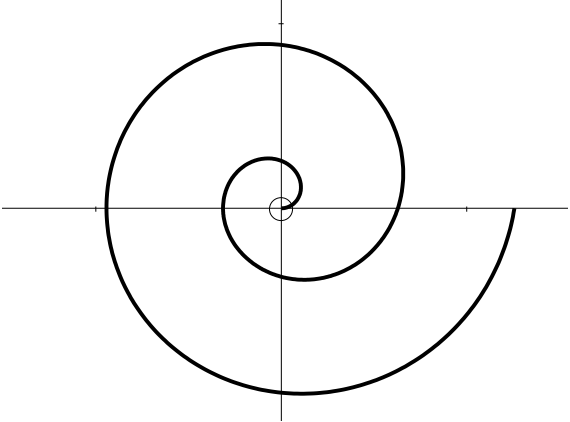
- h For a *genuine* misreading (of numbers or symbols) which is such that the object and the difficulty of the question remain unaltered, mark according to the scheme but following through from the candidate's data. A penalty is then applied; 1 mark is generally appropriate, though this may differ for some units. This is achieved by withholding one A mark in the question.

Note that a miscopy of the candidate's own working is not a misread but an accuracy error.



Question			Answer	Marks	Guidance	
1	(a)	(i)		<p>B1</p> <p>B1</p> <p>[2]</p>	<p>Correct general shape (not multiple-valued, not straight, negative gradient throughout) relative to axes</p> <p>Dependent on first B1.</p> <p>Reasonably vertical at ends.</p> <p>Correct domain (labelled at -1 and 1)</p> <p>Correct range (labelled at <math>\pi</math>)</p> <p>Correct y-intercept (labelled at <math>\pi/2</math>)</p> <p>SC B1B0 for a fully correct curve in <math>[-1, 1] \times [0, \pi]</math> but multiple-valued</p>	
1	(a)	(ii)	$\cos y = x \Rightarrow -\sin y \frac{dy}{dx} = 1$ $\Rightarrow \frac{dy}{dx} = \frac{-1}{\sin y}$ $\sin^2 y + \cos^2 y = 1 \Rightarrow \sin y = (\pm)\sqrt{1-x^2}$ $\Rightarrow \frac{dy}{dx} = \pm \frac{1}{\sqrt{1-x^2}} \text{ or } -\frac{1}{\sqrt{1-x^2}}$ <p>Taking - sign because gradient is negative</p>	<p>M1</p> <p>A1(ag)</p> <p>B1</p> <p>[3]</p>	<p>Differentiating w.r.t. <math>x</math> or <math>y</math></p> <p>Completion w/w with intermediate step</p> <p>Independent of B1 below</p> <p>Validly rejecting + sign.</p> <p>Dependent on A1 above</p>	$\frac{dx}{dy} = -\sin y$ $\frac{dy}{dx} = \pm \frac{1}{\sqrt{1-x^2}} \text{ or } \pm \text{ not considered}$ <p>scores max. 2</p> <p>Or <math>0 \leq y \leq \pi \Rightarrow \sin y \geq 0 \Rightarrow \frac{dy}{dx} \leq 0</math></p> <p>Or <math>f(x)</math> is decreasing</p>

Question			Answer	Marks	Guidance	
1	(a)	(iii)	$f(x) = \arccos x$ $\Rightarrow f'(x) = -1-x^2^{-\frac{1}{2}}$ $\Rightarrow f''(x) = \frac{1}{2} 1-x^2^{-\frac{3}{2}} \times -2x = -x 1-x^2^{-\frac{3}{2}}$ $\Rightarrow f'''(x) = -1-x^2^{-\frac{3}{2}} - x \times -\frac{3}{2} 1-x^2^{-\frac{5}{2}} \times -2x$ $= -1-x^2^{-\frac{3}{2}} - 3x^2 1-x^2^{-\frac{5}{2}}$ $\Rightarrow f(0) = \frac{\pi}{2}$ $f'(0) = -1, f''(0) = 0, f'''(0) = -1$ $\Rightarrow f(x) = \frac{\pi}{2} - x - \frac{x^3}{6} + \dots$	 M1 A1 M1 A1  B1  B1B1	 Derivative in the form $kx 1-x^2^{-\frac{3}{2}}$ o.e. Any correct form www Differentiating $f'' x$ using product or quotient and chain rules. Dep. on 1st M1 Any correct form www  As first term of expansion  $-x$ www, $-\frac{x^3}{6}$ www	 For second derivative  Allow a clear explanation that only the first term contributes to McLaurin expansion for 7/7  Independent of all other marks  Incorrect simplification above loses the last B1
		OR	$f'(x) = -1-x^2^{-\frac{1}{2}} \Rightarrow f(x) = 1 - \frac{1}{2}x^2 \dots$ $\Rightarrow f(x) = \int \left( -1 - \frac{1}{2}x^2 \dots \right) dx = -x - \frac{1}{6}x^3 \dots + c$ $c = \arcsin 0 = \frac{\pi}{2}$	 M2 Using binomial expansion A1A1 $-1, -\frac{1}{2}x^2$  B1B1 www  B1 Correct $c$ as term of expansion	 With $x^2$	
				[7]		

Question			Answer	Marks	Guidance
1	(b)	(i)	$r = \theta + \sin \theta$ $\Rightarrow \frac{dr}{d\theta} = 1 + \cos \theta$ $\cos \theta \geq -1 \Rightarrow \frac{dr}{d\theta} \geq 0, \text{ so } r \text{ increases as } \theta$ <p>increases</p> 	<p>B1</p> <p>B1</p> <p>B1</p> <p>B1</p> <p>[4]</p>	<p><math>\frac{dr}{d\theta} \geq 0</math> stated. Dependent on first B1</p> <p>Do not condone &gt; No wrong statements</p> <p>One complete revolution with <math>r(0) = 0</math> and <math>r(2\pi) \geq r(3\pi/2) \geq r(\pi) \geq r(\pi/2) &gt; 0</math></p> <p>Independent. Condone <math>r(0) &gt; 0</math> for B0B1</p> <p>Correct general shape with two complete revolutions</p>
1	(b)	(ii)	$\text{Area} = \frac{1}{2} \int_0^{\alpha} r^2 d\theta = \frac{1}{2} \int_0^{\alpha} \theta + \sin \theta^2 d\theta$ <p>For small <math>\theta</math>, <math>\sin \theta \approx \theta \Rightarrow r \approx 2\theta</math></p> $\text{Area} \approx \frac{1}{2} \int_0^{\alpha} 2\theta d\theta = \frac{1}{2} \left[ \frac{4}{3} \theta^3 \right]_0^{\alpha}$ $= \frac{2}{3} \alpha^3$	<p>M1</p> <p>M1</p> <p>A1</p> <p>[3]</p>	<p>Forming an integral expression in <math>\theta</math> for the required area</p> <p>Using <math>\sin \theta \approx \theta</math> and a complete method for integrating their expression</p> <p>Condone only omitted limits or <math>\frac{1}{2}</math></p> <p>Dependent on first M1</p>

Question	Answer	Marks	Guidance
2 (a)	$C + jS = ae^{j\theta} + a^2e^{2j\theta} + \dots$ <p>This is a geometric series with <math>r = ae^{j\theta}</math></p> $\text{Sum to infinity} = \frac{ae^{j\theta}}{1 - ae^{j\theta}}$ $= \frac{ae^{j\theta}}{1 - ae^{j\theta}} \times \frac{1 - ae^{-j\theta}}{1 - ae^{-j\theta}}$ $= \frac{ae^{j\theta} - a^2}{1 - ae^{j\theta} - ae^{-j\theta} + a^2}$ $= \frac{a \cos \theta + aj \sin \theta - a^2}{1 - 2a \cos \theta + a^2}$ $= \frac{a \cos \theta - a^2}{1 - 2a \cos \theta + a^2} + \frac{aj \sin \theta}{1 - 2a \cos \theta + a^2}$ $\Rightarrow S = \frac{a \sin \theta}{1 - 2a \cos \theta + a^2}$ <p>and <math>C = \frac{a \cos \theta - a^2}{1 - 2a \cos \theta + a^2}</math></p>	<p>M1</p> <p>M1</p> <p>A1</p> <p>M1*</p> <p>M1</p> <p>M1</p> <p>A1(ag)</p> <p>A1</p> <p>[8]</p>	<p>Forming <math>C + jS</math> as a series of powers</p> <p>Identifying G.P. and attempting sum. Dependent on first M1</p> <p>Correct sum to infinity implies M1M1</p> <p>Multiplying numerator and denominator by <math>1 - ae^{j\theta}</math> o.e.</p> <p>Multiplying out denominator. Dependent on M1*</p> <p>Introducing trig functions. Dependent on M1*</p> <p>Answer given. www which leads to <math>S</math>, e.g. condone sign error in num.</p> <p><b>NB answer space continued (BP)</b></p>

Question			Answer	Marks	Guidance
2	(b)	(i)	$\sqrt{3} + j = 2e^{j\frac{\pi}{6}}$ ; need to rotate by $\frac{\pi}{3}$ so vertices are $2j$ $-\sqrt{3} + j$ $-\sqrt{3} - j$ $-2j$ $\sqrt{3} - j$	 B1 B1 B1 B1 B1  [5]	If vertices not given in form $x + jy$ : B1 for $2e^{j\frac{7\pi}{6}}$ B1 for $2e^{j\frac{\pi}{2}}$ and $2e^{j\frac{3\pi}{2}}$ B1 for $2e^{j\frac{5\pi}{6}}$ and $2e^{j\frac{11\pi}{6}}$ i.e. maximum of 3/5. If B0 scored give SC B2 for five vertices in form $x + jy$ obtained by repeatedly rotating their P by $\frac{\pi}{3}$
2	(b)	(ii)	Vertices are $4e^{j\frac{\pi}{3}} = 2 + 2\sqrt{3}j$ $4e^{j\pi} = -4$ and $4e^{j\frac{5\pi}{3}} = 2 - 2\sqrt{3}j$ Area = $\frac{1}{2} \times 4\sqrt{3} \times 6$ $= 12\sqrt{3}$	M1  A2   B1 [4]	Attempt to square at least one of their vertices in (i)  Three correct in form $x + jy$ (and simplified) and no more  awrt 20.8  Give A1 for any two of these, or all three and no extras in polar form  Dependent on A2 above

Question			Answer	Marks	Guidance	
3	(a)	(i)	Characteristic equation is $(6 - \lambda)(-1 - \lambda) + 12 = 0$ $\Rightarrow \lambda^2 - 5\lambda + 6 = 0$ $\Rightarrow \lambda = 2, 3$ When $\lambda = 2$ , $\begin{pmatrix} 4 & -3 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\Rightarrow 4x - 3y = 0$ $\Rightarrow$ eigenvector is $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ o.e. When $\lambda = 3$ , $\begin{pmatrix} 3 & -3 \\ 4 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\Rightarrow x - y = 0$ $\Rightarrow$ eigenvector is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ o.e.	M1  A1  M1  A1  A1  [5]	Forming characteristic polynomial   At least one equation relating $x$ and $y$	( $\mathbf{A} - \lambda\mathbf{I}$ ) $\mathbf{x} = (\lambda)\mathbf{x}$ M0 below  For either $\lambda = 2$ or $\lambda = 3$
3	(a)	(ii)	$\mathbf{P} = \begin{pmatrix} 3 & 1 \\ 4 & 1 \end{pmatrix}$  $\mathbf{D} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$	B1ft  B1ft  [2]	Do not ft $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ as eigenvector  Columns must correspond	Both fts must be of numerical values  If one matrix diagonal, condone matrices not identified as $\mathbf{P}$ and $\mathbf{D}$

Question			Answer	Marks	Guidance
3	(b)	(i)	$5^3 - 4 \times 5^2 - 3 \times 5 - 10 = 0 \Rightarrow \lambda = 5$ eigenvalue $\lambda^3 - 4\lambda^2 - 3\lambda - 10 = (\lambda - 5)(\lambda^2 + \lambda + 2)$ $\lambda^2 + \lambda + 2 = 0 \Rightarrow \left(\lambda + \frac{1}{2}\right)^2 + \frac{7}{4} = 0 \Rightarrow$ no real roots	B1 M1 A1 A1(ag) [4]	Or showing that $(\lambda - 5)$ is a factor Obtaining quadratic factor Correct quadratic factor Correctly showing a correct quadratic equation has no real roots Two of three terms of quadratic correct e.g. $b^2 - 4ac = 1 - 8$ or correct use of quadratic formula
3	(b)	(ii)	$\mathbf{B} \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} = 5 \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} -10 \\ 5 \\ 20 \end{pmatrix}$ $\mathbf{B}^2 \begin{pmatrix} 4 \\ -2 \\ -8 \end{pmatrix} = 5^2 \begin{pmatrix} 4 \\ -2 \\ -8 \end{pmatrix} = \begin{pmatrix} 100 \\ -50 \\ -200 \end{pmatrix}$ $\mathbf{B} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -20 \\ 10 \\ 40 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -4 \\ 2 \\ 8 \end{pmatrix}$ $\Rightarrow x = -4, y = 2, z = 8$	B1 B1 B2 [4]	Allow $5 \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix}$ isw Allow $25 \begin{pmatrix} 4 \\ -2 \\ -8 \end{pmatrix}$ or $5^2 \begin{pmatrix} 4 \\ -2 \\ -8 \end{pmatrix}$ o.e. Accept vector form Give B1 for two correct unknowns
3	(b)	(iii)	$\text{C-H} \Rightarrow \mathbf{B}^3 - 4\mathbf{B}^2 - 3\mathbf{B} - 10\mathbf{I} = \mathbf{0}$ $\Rightarrow \mathbf{B}^3 = 4\mathbf{B}^2 + 3\mathbf{B} + 10\mathbf{I}$ and $\mathbf{B}^4 = 4\mathbf{B}^3 + 3\mathbf{B}^2 + 10\mathbf{B}$ $= 4(4\mathbf{B}^2 + 3\mathbf{B} + 10\mathbf{I}) + 3\mathbf{B}^2 + 10\mathbf{B}$ $\Rightarrow \mathbf{B}^4 = 19\mathbf{B}^2 + 22\mathbf{B} + 40\mathbf{I}$	M1 M1 A1(ag) [3]	Idea of $\lambda \leftrightarrow \mathbf{B}$ . Condone omitted $\mathbf{I}$ Multiplying by $\mathbf{B}$ and substituting for $\mathbf{B}^3$ Completion Condone use of $\mathbf{M}$ throughout

Question	Answer	Marks	Guidance
4 (i)	$x = \sinh y \Rightarrow x = \frac{e^y - e^{-y}}{2}$ $\Rightarrow e^y - e^{-y} = 2x$ $\Rightarrow e^{2y} - 2xe^y - 1 = 0$ $\Rightarrow e^y - x^2 = 1 + x^2$ $\Rightarrow e^y = x \pm \sqrt{1+x^2}$ $\Rightarrow y = \ln  x \pm \sqrt{1+x^2} $ $x - \sqrt{1+x^2} < 0 \text{ so take } + \text{ sign}$	B1     M1 A1(ag)  B1	$x$ in exponential form     Solving to reach $e^y$ Completion www Validly rejecting negative root. Dependent on A1 above  Allow one slip. Ignore variables. Allow unsimplified $y = \ln  x \pm \sqrt{x^2 + 1} $ A0 e.g. $e^y > 0$ ; $e^y \geq 0$ B0
	<b>OR</b> $\ln  x + \sqrt{1+x^2}  = \ln  \sinh y + \sqrt{1+\sinh^2 y} $ $= \ln  \sinh y + \cosh y $ $= \ln(e^y)$ $= y$	M1  B1 B1  A1	Explanation why + is taken  Completion www  e.g. $\sinh y - \cosh y < 0$
	$\Rightarrow \frac{dy}{dx} = \frac{1}{x + \sqrt{1+x^2}} \times \frac{d}{dx} (x + \sqrt{1+x^2})$ $= \frac{1}{x + \sqrt{1+x^2}} \times \left( 1 + \frac{x}{\sqrt{1+x^2}} \right)$ $= \frac{1}{x + \sqrt{1+x^2}} \times \left( \frac{\sqrt{1+x^2} + x}{\sqrt{1+x^2}} \right) \quad (*)$ $= \frac{1}{\sqrt{1+x^2}}$	M1  B1 A1   A1(ag)  <b>[8]</b>	Attempting $\frac{1}{u} \times \frac{du}{dx}$  $\frac{d}{dx} (x + \sqrt{1+x^2}) = 1 + \frac{x}{\sqrt{1+x^2}}$ o.e. Any correct form of $\frac{dy}{dx}$ in terms of $x$  Obtained www with valid intermediate step, e.g. (*)  <b>NB answer space continued (BP)</b>



Question	Answer	Marks	Guidance
4 (ii)	$\int \frac{1}{\sqrt{25+4x^2}} dx = \frac{1}{2} \int \frac{1}{\sqrt{\frac{25}{4} + x^2}} dx$ $= \frac{1}{2} \ln \left( x + \sqrt{x^2 + \frac{25}{4}} \right) + c$	M1 A1 A1 <b>[3]</b>	arsinh $kx$ or $\ln kx + \sqrt{k^2x^2 + \dots}$ $\frac{1}{2} \operatorname{arsinh} \frac{2x}{5}$ or $\ln \left( \frac{2x}{5} + \sqrt{1 + \frac{4x^2}{25}} \right)$ o.e. Fully correct in logarithmic form Condone omitted $c$
4 (iii)	$2x = 5 \sinh u \Rightarrow \frac{dx}{du} = \frac{5}{2} \cosh u$ $\int \sqrt{25+4x^2} dx = \int \sqrt{25+25\sinh^2 u} \times \frac{5}{2} \cosh u du$ $= \int \frac{25}{2} \cosh^2 u du$ $= \int \left( \frac{25}{4} \cosh 2u + \frac{25}{4} \right) du$ $= \frac{25}{8} \sinh 2u + \frac{25}{4} u + c$ $= \frac{25}{4} \sinh u \cosh u + \frac{25}{4} u + c$ $= \frac{25}{4} \times \frac{2x}{5} \times \sqrt{1 + \frac{4x^2}{25}} + \frac{25}{4} \operatorname{arsinh} \frac{2x}{5} + c$ $= \frac{25}{4} \left( \ln \left( \frac{2x}{5} + \sqrt{1 + \frac{4x^2}{25}} \right) + \frac{2x}{5} \sqrt{1 + \frac{4x^2}{25}} \right) + c$	M1 A1 M1* A2 M1 A1(ag) <b>[7]</b>	Finding $\frac{dx}{du}$ and complete substitution Substituting for all elements correctly Simplifying an expression of the form $k \cosh^2 u$ to an integrable form Any correct form. Condone omitted $c$ Give A2ft for $\frac{k}{4} \sinh 2u + \frac{ku}{2}$ Give A1ft A0 for one error Using double “angle” formula Dependent on M1* Completion wwww with convincing intermediate step Condone “upside-down” substitution for $dx$ e.g. $\frac{25}{8} e^{2u} + \frac{25}{4} + \frac{25}{8} e^{-2u}$ e.g. $\frac{25}{16} e^{2u} + \frac{25}{4} u + \frac{25}{16} e^{-2u} + c$ Using exponential definition of $\sinh 2u$ and substituting for $u$ scores this M1 if an expression with a constant denominator is found validly e.g. reversing terms <b>NB answer space continued (BP)</b>

## 4756 Further Methods for Advanced Mathematics (FP2)

### General Comments:

The overall performance of candidates was comparable with previous series. The vast majority of candidates displayed sound knowledge of standard results and techniques; very few scored 20 marks or fewer, and about one-third scored 60 marks or more. Question 3 (matrices) was the best-done question, followed by Question 2 (complex numbers) and then Question 1 (calculus and polar co-ordinates) and Question 4 (hyperbolic functions).

Presentation was generally good and most scripts were easy to follow, although a few made extensive use of supplementary sheets. Two graphs were required and many candidates drew these carefully, to an appropriate size, and with appropriately-labelled axes. There was very little evidence of time trouble.

Candidates could have done even better if they had:

- resolved the ambiguities of sign in Q1(a)(ii) (especially) and Q4(i);
- been more precise with their use of language, for example, in Q1(b)(i), “gradient is positive” is not true for all values of  $\theta$ ;
- used simpler methods if possible, for example, in Q1(b)(ii);
- understood better what an eigenvector is, rather than just how to find one (Q3(b)(ii)).

### Comments on Individual Questions:

Question No.1 (calculus and polar co-ordinates).

Part (a) was about the inverse cosine function. In (i), the graph of  $y = \arccos x$  was required. Most candidates had a good idea of the general shape but only a small proportion gained both marks. Candidates were expected to label the axes at  $-1$ ,  $1$ ,  $\pi/2$  and  $\pi$  and many omitted at least one of these; many also found difficulty in representing the undefined gradient at  $\pm 1$ . There were many multiple-valued functions, which were given some credit as long as critical points were labelled.

In (ii) candidates were required to differentiate the function. The vast majority could score 2/3 very efficiently, although a few omitted the trigonometric identity required to score the second mark. Very few observed that the gradient was negative, which resolved the ambiguity of sign.

Then in (iii) a Maclaurin series for  $\arccos x$  was required. One possible method was to expand the answer to (ii) by the Binomial Theorem and integrate the first few terms, but few chose that route and instead differentiated the expression in (ii) to obtain the second, and then the third, derivative. A substantial number managed this perfectly well, but others lost the minus sign from (ii), and for many it proved too difficult to manage the combination of the product and chain rules required to obtain the third derivative. Most candidates knew how to obtain a Maclaurin series from their results, although some gave the first term as 90, rather than  $\pi/2$ .

Part (b) was about the polar curve  $r = \theta + \sin\theta$ . In (i), the vast majority were able to differentiate  $r$  with respect to  $\theta$ , although on a few scripts  $\theta$  disappeared on differentiation. The explanation of why  $r$  increases as  $\theta$  increases was often insufficiently precise, as mentioned above. The graph was well done, with many scoring both marks.

In (ii), we expected candidates to substitute  $\theta$  for  $\sin\theta$ , and then integrate a multiple of  $\theta^2$ . Many candidates did this, scoring all three marks very efficiently, but a substantial number either attempted to integrate  $\theta^2 + 2\theta\sin\theta + \sin^2\theta$  by parts and via double angle formulae, and then

either gave up, or substituted  $\theta$  for  $\sin\theta$  at some later stage, or substituted  $\sin\theta$  for  $\theta$  at the beginning. Both methods were rarely completely successful: if candidates ended up with a term in  $\cos\theta$  in the result of their integration, they generally did not know how to replace it by a valid small angle approximation.

Answers:

(a)(i) graph, (ii) given answer, (iii)  $\frac{\pi}{2} - x - \frac{x^3}{6}$ ; (b)(i)  $\frac{dr}{dq} = 1 + \cos q$  which is never negative; graph (ii)  $\frac{2}{3}a^3$ .

Question No.2 (complex numbers)

Part (a) was well done. Most candidates were able to form a geometric series and sum it correctly; a few produced the sum to  $n$  terms, rather than the sum to infinity. The big hurdle, that of “realising” the denominator, was crossed by a large number of candidates, many of which were able to pursue their solution to a fully successful conclusion. A few multiplied  $ae^{jq}$  by  $ae^{-jq}$  and obtained 1, and there were other minor errors, but this question was done rather better than similar questions in previous series.

Part (b) then explored some complex number geometry, and was an excellent source of marks for many candidates. Most managed (i) very efficiently, usually finding the other vertices of the hexagon by expressing  $\sqrt{3} + j$  in exponential form or equivalent, and then repeatedly adding  $\frac{\pi}{3}$  to the argument, or by using symmetry. Partial credit was given to candidates who did not express their final answers in the required form. A few were determined to find the sixth roots of  $\sqrt{3} + j$ .

In (ii) they had to square these complex numbers to form a new figure, and find its area. Most managed this well but some errors crept in; a few squared the real and imaginary parts separately, while others made slips which produced more than three distinct complex numbers and ended up finding areas of irregular pentagons and other shapes, rather than the correct isosceles triangle.

Answers:

(a) S given,  $C = \frac{a \cos q - a^2}{1 - 2a \cos q + a^2}$ ; (b)(i)  $2j, -\sqrt{3} + j, -\sqrt{3} - j, -2j, \sqrt{3} - j$ ; (ii)  $2 \pm 2\sqrt{3}j, -4; 12\sqrt{3}$ .

Question No.3 (matrices)

Part (a)(i), which required candidates to find eigenvalues and eigenvectors for a  $2 \times 2$  matrix, was extremely well done, with most candidates achieving full marks. Errors, where they occurred, included making slips in forming or solving the characteristic equation (often leading to

6 and  $-1$  as eigenvalues) and giving an eigenvector as  $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$  when faced with the equation  $4x -$

$3y = 0$ . Part (ii) was as well done as part (i), although “eigenvectors” of  $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$  were not followed

through as they give a singular matrix for **P**.

In part (b), (i) was very well handled, with, again, most candidates achieving full marks. There were very few errors in showing that  $\lambda = 5$  was a root of the cubic characteristic equation, and the quadratic factor was obtained very efficiently. Showing that the associated quadratic

equation had no real roots was usually accomplished via the discriminant; we condoned this being referred to as the “determinant” (among other terms).

Part (ii) was less well done. This part required candidates to consider what eigenvectors do, and, although there were many efficient solutions, many candidates attempted to find the elements of the matrix B, or omitted the part altogether: about a quarter of candidates failed to score at all. A substantial number of otherwise successful candidates misread the second instruction and tried

$$\text{to find } \mathbf{B}^2 = \begin{pmatrix} 2 & 0 \\ 1 & 4 \end{pmatrix}$$

By contrast, part (iii), using the Cayley-Hamilton theorem, was extremely well done, with a very high proportion of well-expressed and fully correct solutions.

Answers:

(a)(i) eigenvalues 2 and 3; corresponding eigenvectors  $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$  and  $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ ; (ii)  $\mathbf{P} = \begin{pmatrix} 1 & 3 \\ 1 & 4 \end{pmatrix}$  and

$$\mathbf{D} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}; \text{ (b)(i) given answers; (ii) } \begin{pmatrix} 10 & 100 \\ 5 & -50 \\ 20 & -200 \end{pmatrix}; x = -4, y = 2 \text{ and } z = 8; \text{ (iii) given answer.}$$

Question No.4 (hyperbolic functions)

Part (i) required candidates to obtain the logarithmic form of arsinh, and then differentiate it. The vast majority knew the method required for the first item and did it efficiently, although as in Q1(a)(ii), not many were able to resolve the ambiguity of sign and there were many spurious arguments involving the sum or the product of the “roots” (one of which did not exist as a real number). Then many were able to differentiate the given expression, but far fewer were able to

show convincingly that their result was equivalent to the required  $\frac{1}{\sqrt{1+x^2}}$ . Some candidates

ignored the instruction to “differentiate (\*)” and went back to the hyperbolic functions, often because they could not perform the last step and show that their derivative was equivalent to the given result; if candidates had deleted otherwise good work to do this, some credit was given if the work could be read.

Part (ii) was very well done. A variety of (correct) logarithmic forms were given, although some left their answer as an arsinh and others omitted the  $\frac{1}{2}$ .

Part (iii) was a challenging final part which produced a good spread of marks. Most candidates used the given substitution although a few preferred their own. One or two thought that

$\sqrt{25+4x^2} = 5+2x$  and then went ahead and used the hyperbolic substitution anyway. A few performed the change of variable from dx to du “upside down”, so that cosh u ended up in the denominator (and they had a constant to integrate), but many were able to reach an expression involving  $\cosh^2 u$  and many of those used a double “angle” formula to produce an expression they could integrate; fewer converted to exponential form. Having obtained an expression of the form  $\frac{k}{4} \sinh 2u + \frac{ku}{2}$ , many could not make further progress and write their answer in terms of x;

they often just copied the given answer. But there were a substantial number of candidates who could make this last step and obtained full, or nearly full, marks. One or two candidates

attempted to reintroduce  $x$  by using the exponential form of  $\sinh 2u$ : this often filled pages and was never fully successful.

Answers:

(i) given answers; (ii)  $\frac{1}{2} \ln \left( x + \sqrt{x^2 + \frac{25}{4}} \right)$ ; (iii) given answer.