OXFORD CAMBRIDGE AND RSA EXAMINATIONS

Advanced Subsidiary General Certificate of Education
Advanced General Certificate of Education

MEI STRUCTURED MATHEMATICS 4753/1

Methods for Advanced Mathematics (C3)

Thursday **8 JUNE 2006** Morning 1 hour 30 minutes

Additional materials:
8 page answer booklet
Graph paper
MEI Examination Formulae and Tables (MF2)

TIME 1 hour 30 minutes

INSTRUCTIONS TO CANDIDATES

- Write your name, centre number and candidate number in the spaces provided on the answer booklet.
- Answer **all** the questions.
- You are permitted to use a graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets [ ] at the end of each question or part question.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.
- The total number of marks for this paper is 72.

This question paper consists of 5 printed pages and 3 blank pages.

Section A (36 marks)

1 Solve the equation $|3x - 2| = x$. [3]

2 Show that $\int_0^{\frac{1}{6}\pi} x \sin 2x \, dx = \frac{3\sqrt{3} - \pi}{24}$. [6]

3 Fig. 3 shows the curve defined by the equation $y = \arcsin(x - 1)$, for $0 \leqslant x \leqslant 2$.

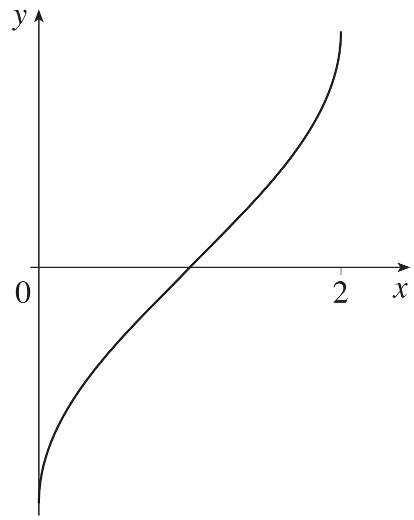


Fig. 3

(i) Find x in terms of y, and show that $\frac{dx}{dy} = \cos y$. [3]

(ii) Hence find the exact gradient of the curve at the point where $x = 1.5$. [4]

4 Fig. 4 is a diagram of a garden pond.

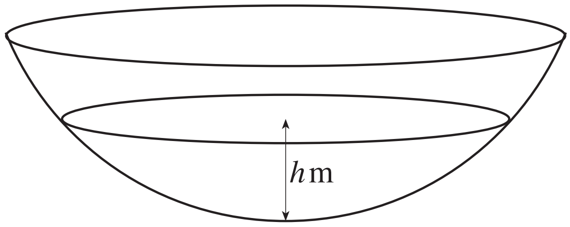


Fig. 4

The volume $V\,\text{m}^3$ of water in the pond when the depth is h metres is given by

$$V = \tfrac{1}{3}\pi h^2(3 - h).$$

(i) Find $\dfrac{\mathrm{d}V}{\mathrm{d}h}$. [2]

Water is poured into the pond at the rate of $0.02\,\text{m}^3$ per minute.

(ii) Find the value of $\dfrac{\mathrm{d}h}{\mathrm{d}t}$ when $h = 0.4$. [4]

5 Positive integers a, b and c are said to form a Pythagorean triple if $a^2 + b^2 = c^2$.

(i) Given that t is an integer greater than 1, show that $2t$, $t^2 - 1$ and $t^2 + 1$ form a Pythagorean triple. [3]

(ii) The two smallest integers of a Pythagorean triple are 20 and 21. Find the third integer.

Use this triple to show that not all Pythagorean triples can be expressed in the form $2t$, $t^2 - 1$ and $t^2 + 1$. [3]

6 The mass M kg of a radioactive material is modelled by the equation

$$M = M_0 \mathrm{e}^{-kt},$$

where M_0 is the initial mass, t is the time in years, and k is a constant which measures the rate of radioactive decay.

(i) Sketch the graph of M against t. [2]

(ii) For Carbon 14, $k = 0.000\,121$. Verify that after 5730 years the mass M has reduced to approximately half the initial mass. [2]

The half-life of a radioactive material is the time taken for its mass to reduce to exactly half the initial mass.

(iii) Show that, in general, the half-life T is given by $T = \dfrac{\ln 2}{k}$. [3]

(iv) Hence find the half-life of Plutonium 239, given that for this material $k = 2.88 \times 10^{-5}$. [1]

[Turn over

Section B (36 marks)

7 Fig. 7 shows the curve $y = \dfrac{x^2 + 3}{x - 1}$. It has a minimum at the point P. The line l is an asymptote to the curve.

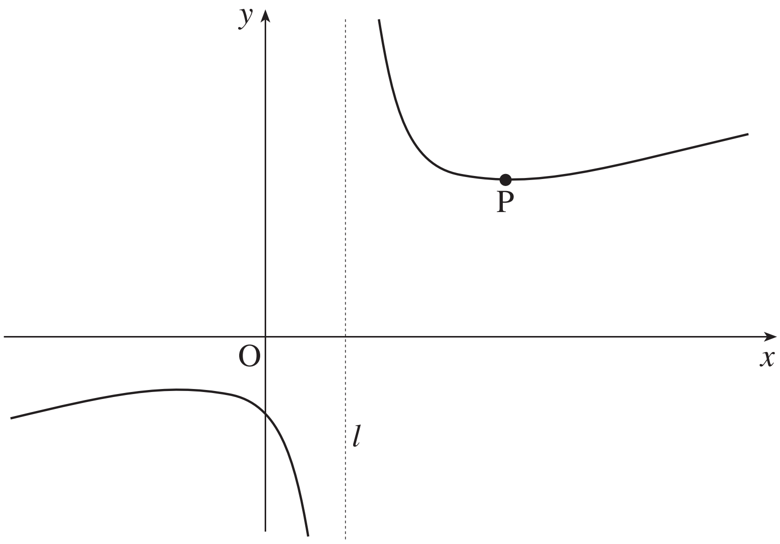


Fig. 7

(i) Write down the equation of the asymptote l. [1]

(ii) Find the coordinates of P. [6]

(iii) Using the substitution $u = x - 1$, show that the area of the region enclosed by the x-axis, the curve and the lines $x = 2$ and $x = 3$ is given by

$$\int_1^2 \left(u + 2 + \frac{4}{u}\right) \mathrm{d}u.$$

Evaluate this area exactly. [7]

(iv) Another curve is defined by the equation $\mathrm{e}^y = \dfrac{x^2 + 3}{x - 1}$. Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x and y by differentiating implicitly. Hence find the gradient of this curve at the point where $x = 2$. [4]

8 Fig. 8 shows part of the curve $y = f(x)$, where $f(x) = e^{-\frac{1}{5}x}\sin x$, for all x.

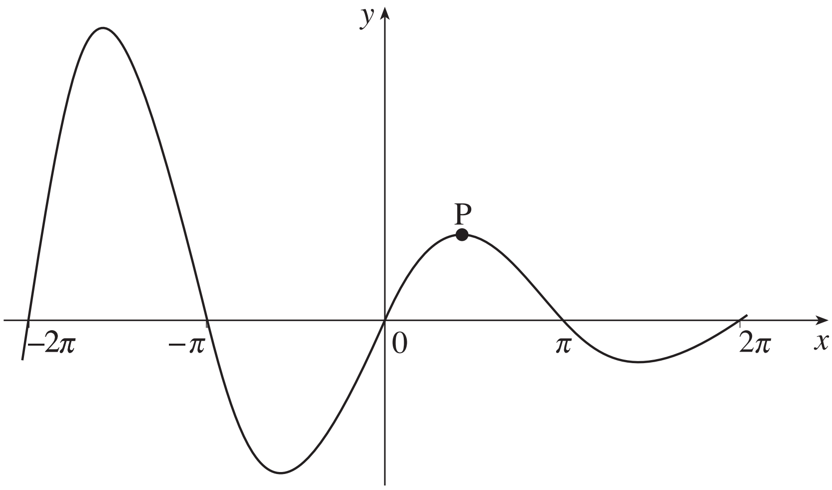


Fig. 8

(i) Sketch the graphs of

(A) $y = f(2x)$,

(B) $y = f(x + \pi)$. [4]

(ii) Show that the x-coordinate of the turning point P satisfies the equation $\tan x = 5$.

Hence find the coordinates of P. [6]

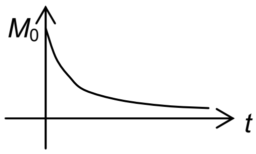
(iii) Show that $f(x + \pi) = -e^{-\frac{1}{5}\pi} f(x)$. Hence, using the substitution $u = x - \pi$, show that

$$\int_{\pi}^{2\pi} f(x)\,dx = -e^{-\frac{1}{5}\pi}\int_{0}^{\pi} f(u)\,du.$$

Interpret this result graphically. [You should *not* attempt to integrate $f(x)$.] [8]

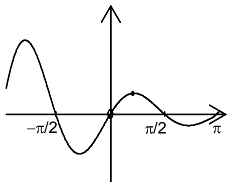
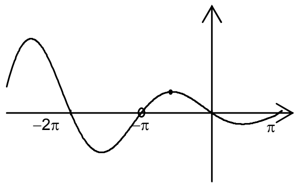
Mark Scheme 4753
June 2006

1 $\lvert 3x-2\rvert = x$ $\Rightarrow$ $3x - 2 = x \Rightarrow 2x = 2 \Rightarrow x = 1$ or $2 - 3x = x \Rightarrow 2 = 4x \Rightarrow x = ½$ *or* $(3x - 2)^2 = x^2$ $\Rightarrow 8x^2 - 12x + 4 = 0 \Rightarrow 2x^2 - 3x + 1 = 0$ $\Rightarrow (x - 1)(2x - 1) = 0,$ $\Rightarrow x = 1, ½$	B1 M1 A1 M1 A1 A1 [3]	$x = 1$ solving correct quadratic
2 let $u = x$, $dv/dx = \sin 2x \Rightarrow v = -½\cos 2x$ $\Rightarrow \int_0^{\pi/6} x\sin 2x dx = \left[x.-\frac{1}{2}\cos 2x\right]_0^{\pi/6} + \int_0^{\pi/6}\frac{1}{2}\cos 2x.1.dx$ $= \frac{\pi}{6}.-\frac{1}{2}\cos\frac{\pi}{3} - 0 + \left[\frac{1}{4}\sin 2x\right]_0^{\frac{\pi}{6}}$ $= -\frac{\pi}{24} + \frac{\sqrt{3}}{8}$ $= \frac{3\sqrt{3}-\pi}{24}$ *	M1 A1 B1ft M1 B1 E1 [6]	parts with $u = x$, $dv/dx = \sin 2x$ $... + \left[\frac{1}{4}\sin 2x\right]_0^{\frac{\pi}{6}}$ substituting limits $\cos \pi/3 = ½$, $\sin \pi/3 = \sqrt{3}/2$ soi www
3 (i) $x - 1 = \sin y$ $\Rightarrow$ $x = 1 + \sin y$ $\Rightarrow$ $dx/dy = \cos y$ **(ii)** When $x = 1.5$, $y = \arcsin(0.5) = \pi/6$ $\frac{dy}{dx} = \frac{1}{\cos y}$ $= \frac{1}{\cos \pi/6}$ $= 2/\sqrt{3}$	M1 A1 E1 M1 A1 M1 A1 [7]	 www condone 30° or 0.52 or better or $\frac{dy}{dx} = \frac{1}{\sqrt{1-(x-1)^2}}$ or equivalent, but must be exact
4(i) $V = \pi h^2 - \frac{1}{3}\pi h^3$ $\Rightarrow \frac{dV}{dh} = 2\pi h - \pi h^2$ **(ii)** $\frac{dV}{dt} = 0.02$ $\frac{dV}{dt} = \frac{dV}{dh}.\frac{dh}{dt}$ $\Rightarrow \frac{dh}{dt} = \frac{0.02}{dV/dh} = \frac{0.02}{2\pi h - \pi h^2}$ When $h = 0.4$, $\Rightarrow \frac{dh}{dt} = \frac{0.02}{0.8\pi - 0.16\pi} = 0.0099\,\mathrm{m/min}$	M1 A1 B1 M1 M1dep A1cao [6]	expanding brackets (correctly) or product rule oe soi $\frac{dV}{dt} = \frac{dV}{dh}.\frac{dh}{dt}$ oe substituting $h = 0.4$ into their $\frac{dV}{dh}$ and $\frac{dV}{dt} = 0.02$ 0.01 or better or $1/32\pi$

5(i) $a^2 + b^2 = (2t)^2 + (t^2 - 1)^2$ $= 4t^2 + t^4 - 2t^2 + 1$ $= t^4 + 2t^2 + 1$ $= (t^2 + 1)^2 = c^2$	M1 M1 E1	substituting for a, b and c in terms of t Expanding brackets correctly www
(ii) $c = \sqrt{(20^2 + 21^2)} = 29$ For example: $2t = 20 \Rightarrow t = 10$ $\Rightarrow$ $t^2 - 1 = 99$ which is not consistent with 21	B1 M1 E1 [6]	 Attempt to find t Any valid argument or E2 'none of 20, 21, 29 differ by two'.
6 (i) 	B1 B1	Correct shape Passes through $(0, M_0)$
(ii) $\frac{M}{M_0} = e^{-0.000121 \times 5730} = e^{-0.6933...} \approx \frac{1}{2}$	M1 E1	substituting $k = -0.00121$ and $t = 5730$ into equation (or ln eqn) showing that $M \approx ½ M_0$
(iii) $\frac{M}{M_0} = e^{-kT} = \frac{1}{2}$ $\Rightarrow \ln\frac{1}{2} = -kT$ $\Rightarrow \ln 2 = kT$ $\Rightarrow T = \frac{\ln 2}{k}$ *	M1 M1 E1	substituting $M/M_0 = ½$ into equation (oe) taking lns correctly
(iv) $T = \frac{\ln 2}{2.88 \times 10^{-5}} \approx 24\,000$ years	B1 [8]	24 000 or better

Section B

7(i) $x = 1$	B1 [1]	
(ii) $\frac{dy}{dx}=\frac{(x-1)2x-(x^2+3).1}{(x-1)^2}$ $=\frac{2x^2-2x-x^2-3}{(x-1)^2}$ $=\frac{x^2-2x-3}{(x-1)^2}$ $dy/dx = 0$ when $x^2 - 2x - 3 = 0$ $\Rightarrow (x-3)(x+1)=0$ $\Rightarrow x = 3$ or -1 When $x = 3$, $y = (9+3)/2 = 6$ So P is (3, 6)	M1 A1 M1 M1 A1 B1ft [6]	Quotient rule correct expression their numerator = 0 solving quadratic by any valid method $x = 3$ from correct working $y = 6$
(iii) Area $=\int_2^3 \frac{x^2+3}{x-1}dx$ $u = x - 1 \Rightarrow du/dx = 1,\ du = dx$ When $x = 2$, $u = 1$; when $x = 3$, $u = 2$ $=\int_1^2\frac{(u+1)^2+3}{u}du$ $=\int_1^2\frac{u^2+2u+4}{u}du$ $=\int_1^2(u+2+\frac{4}{u})du$ * $=\left[\frac{1}{2}u^2+2u+4\ln u\right]_1^2$ $= (2 + 4 + 4\ln 2) - (\frac{1}{2} + 2 + 4\ln 1)$ $= 3\frac{1}{2} + 4\ln 2$	M1 B1 B1 E1 B1 M1 A1cao [7]	Correct integral and limits Limits changed, and substituting $dx = du$ substituting $\frac{(u+1)^2+3}{u}$ www $[\frac{1}{2}u^2 + 2u + 4\ln u]$ substituting correct limits
(iv) $e^y=\frac{x^2+3}{x-1}$ $\Rightarrow e^y\frac{dy}{dx}=\frac{x^2-2x-3}{(x-1)^2}$ $\Rightarrow \frac{dy}{dx}=e^{-y}\frac{x^2-2x-3}{(x-1)^2}$ When $x = 2$, $e^y = 7 \Rightarrow$ $\Rightarrow dy/dx = \frac{1}{7}.\frac{4-4-3}{1} = -\frac{3}{7}$	M1 A1ft B1 A1cao [4]	$e^y dy/dx$ = their $f'(x)$ or $xe^y - e^y = x^2 + 3$ $\Rightarrow e^y + xe^y\frac{dy}{dx} - e^y\frac{dy}{dx} = 2x$ $\Rightarrow \frac{dy}{dx}=\frac{2x-e^y}{e^y(x-1)}$ $y = \ln 7$ or 1.95… or $e^y = 7$ or $\frac{dy}{dx}=\frac{4-7}{7(2-1)}=-\frac{3}{7}$ or −0.43 or better

8 (i) (*A*)  (*B*) 	B1 B1 M1 A1 [4]	Zeros shown every π/2. Correct shape, from −π to π Translated in *x*-direction π to the left
(ii) $f'(x) = -\frac{1}{5}e^{-\frac{1}{5}x}\sin x + e^{-\frac{1}{5}x}\cos x$ $f'(x) = 0$ when $-\frac{1}{5}e^{-\frac{1}{5}x}\sin x + e^{-\frac{1}{5}x}\cos x = 0$ $\Rightarrow \frac{1}{5}e^{-\frac{1}{5}x}(-\sin x + 5\cos x) = 0$ $\Rightarrow \sin x = 5\cos x$ $\Rightarrow \frac{\sin x}{\cos x} = 5$ $\Rightarrow \tan x = 5$* $\Rightarrow x = 1.37(34\ldots)$ $\Rightarrow y = 0.75$ or $0.74(5\ldots)$	B1 B1 M1 E1 B1 B1 [6]	$e^{-\frac{1}{5}x}\cos x$ $\ldots -\frac{1}{5}e^{-\frac{1}{5}x}\sin x$ dividing by $e^{-\frac{1}{5}x}$ www 1.4 or better, must be in radians 0.75 or better
(iii) $f(x+\pi) = e^{-\frac{1}{5}(x+\pi)}\sin(x+\pi)$ $= e^{-\frac{1}{5}x}e^{-\frac{1}{5}\pi}\sin(x+\pi)$ $= -e^{-\frac{1}{5}x}e^{-\frac{1}{5}\pi}\sin x$ $= -e^{-\frac{1}{5}\pi}f(x)$* $\int_{\pi}^{2\pi} f(x)dx$ let $u = x - \pi$, $du = dx$ $= \int_0^{\pi} f(u+\pi)du$ $= \int_0^{\pi} -e^{-\frac{1}{5}\pi}f(u)du$ $= -e^{-\frac{1}{5}\pi}\int_0^{\pi} f(u)du$* Area enclosed between π and 2π $= (-)e^{-\frac{1}{5}\pi} \times$ area between 0 and π.	M1 A1 A1 E1 B1 B1dep E1 B1 [8]	$e^{-\frac{1}{5}(x+\pi)} = e^{-\frac{1}{5}x}.e^{-\frac{1}{5}\pi}$ $\sin(x+\pi) = -\sin x$ www $\int f(u+\pi)du$ limits changed using above result or repeating work or multiplied by 0.53 or better

4753 – Methods for Advanced Mathematics (C3) (Written Examination)

General Comments

The paper attracted the usual range of responses, with many scripts scoring 60 or more, and a substantial 'tail' of weak candidates scoring less than 20. There was some excellent work seen, with well presented, accurate solutions. In general, it appears that the standard calculus techniques are well understood by candidates, but some of the more peripheral topics in the syllabus, like modulus notation and inverse trigonometric functions, are less secure.

Algebra, as usual, is a stumbling block: for example, basic errors in expanding brackets were seen in questions 5 and 7(ii). Despite the fact that most candidates for this summer paper will have done substantially more work on trigonometry from studying the C4 module, the exact values of trigonometric ratios for 60° and 30°, required for questions 2 and 3, were sometimes not known.

It would be worth mentioning to candidates for this paper that although graph paper is supplied on demand, it is hard to conceive of a question where it should be used – graphs are almost always better drawn as sketches on their answer paper.

Question 8 was found be the hardest question, so it is difficult to know if weak attempts to this could be ascribed to a lack of time. However, all but a few candidates seemed to have sufficient time to attempt all the questions.

Comments on Individual Questions

Section A

1 Knowledge of the modulus notation was quite variable and centre-dependent. Algebraic errors abounded and x=1 was quite often the only solution. 4x=2 rather too frequently led to x=2! Also $|x| = 1$ or $x = \pm 1$ were common errors. Candidates who mis-interpreted the question as an inequality gained no marks.

2 Nearly all the candidates recognised this question as an integration by parts, with $u = x$ and $\mathrm{d}v/\mathrm{d}x = \sin 2x$. However, only strong candidates managed to derive the exact result. Quite a few candidates made errors in v such as wrong sign, omitted factor of ½, etc. Many candidates failed to substitute exact values for $\sin \pi/3$ and $\cos \pi/3$, and verified the given result numerically.

3 This question received a mixed response. Many candidates gained the first three marks without difficulty, but the misconception that $\sin^{-1}x = 1/\sin x$ was seen quite frequently. In part (ii), many found $y = \pi/6$ (30° was condoned here), but either gave the answer as $\cos y$ or failed to give an exact result.

4 Weaker candidates failed to differentiate V correctly, and a surprising number opted to use the product rule rather than expanding the brackets. In part (ii), $\mathrm{d}V/\mathrm{d}t = 0.02$ was an easy mark, and most candidates wrote a successful form of the chain rule and substituted correctly.

5 Part (i) was answered competently by many candidates, though weaker candidates failed to expand $(t^2 \pm 1)^2$ correctly, or just verified the result for $t = 1$. In part (ii), virtually all worked out $c = 29$ correctly, and there were quite a few who argued the proof by contradiction convincingly. A common source of error here was to start by taking $t = 20$ instead of $2t = 20$.

6 In part (i), most of the sketches were of the correct shape, but many omitted the intercept M_0 on the vertical axis. Part (ii) was well answered. In part (iii), many candidates failed to substitute $M/M_0 = ½$, for example starting $M = ½\, M_0 e^{-kt}$. Using a particular value for M_0, such as 1, gained 1 out of 3 marks, even if they took $M_0 = 14$, which was a surprisingly popular choice! Part (iv) was a mark for all and sundry.

Section B

7 Even the weakest candidates scored marks in this question, and there were many strong solutions

(i) This was well answered; errors included $y = 1$.

(ii) The quotient rule was well known and usually correct, and sound candidates achieved these 6 marks without difficulty. However, it was disappointing to see so many candidates making algebraic slips with the brackets in the numerator, and having to 'fudge' the resulting quadratic.

(iii) Well prepared candidates derived the stated transformed integral without difficulty. There was some sloppy notation – missing d*u*'s or d*x*'s, limits inconsistent with *x* or *u*, omitted integral signs, etc. Most of this was condoned, but to achieve full marks, students needed to state that $du/dx = 1$ (or $du = dx$). The integration was very poorly done, with the majority of candidates failing to integrate the $4/u$ term as $4 \ln u$.

(iv) This was a more demanding test. The implicit differentiation was mixed, and some left dy/dx as a 'double-decker' fraction, which lost an A mark. Some then substituted $y = 7$ instead of $\ln 7$ into the derivative. Some tried logarithmic differentiation, but then differentiated $\ln(x^2 + 3)$ incorrectly as $1/(x^2 + 3)$.

8 Solutions may have suffered from lack of time in some cases. However, this was clearly the most demanding question on the paper, and few candidates scored well.

(i) Some sketch graphs were very hard to decipher, especially when all three graphs were shown super-imposed. In general, the translation in (B) was perhaps better answered than the stretch in (A). We wanted to see the diminishing amplitudes preserved, but allowed some leeway here. Domain errors were common.

(ii) The product rule was quite well answered, and many factored out the $e^{-x/5}$ term. However, only good candidates derived $\tan x = 5$ successfully – some resorted to verifying numerically. The calculation of *x* and *y* should have been straightforward, but many used degrees for *x* instead of radians. It should be noted that $x = \arctan 5$ was not allowed – we require candidates to evaluate this.

(iii) This was by far the least successful question on the paper. There were many attempts to 'fudge' the result given, mainly because candidates made errors with the $\sin(x + \pi)$ part – say $\sin x + \sin \pi$, for example – and could not see where the negative sign came from. The final 4 marks were rarely achieved, with lots of unconvincing work based on transforming the graph. Candidates needed to show $\int f(u+\pi)du$ with evidence of transformed limits to gain the marks. The interpretation of the results required candidates to relate this to the areas of successive sections of the curve.