OCR
RECOGNISING ACHIEVEMENT

ADVANCED GCE **4753/01**

MATHEMATICS (MEI)

Methods for Advanced Mathematics (C3)

FRIDAY 11 JANUARY 2008 Morning

Time: 1 hour 30 minutes

Additional materials: Answer Booklet (8 pages)
Graph paper
MEI Examination Formulae and Tables (MF2)

INSTRUCTIONS TO CANDIDATES

- Write your name in capital letters, your Centre Number and Candidate Number in the spaces provided on the Answer Booklet.
- Read each question carefully and make sure you know what you have to do before starting your answer.
- Answer **all** the questions.
- You are permitted to use a graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets [ ] at the end of each question or part question.
- The total number of marks for this paper is 72.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.

This document consists of **4** printed pages.

Section A (36 marks)

1 Differentiate $\sqrt[3]{1+6x^2}$. [4]

2 The functions $\mathrm{f}(x)$ and $\mathrm{g}(x)$ are defined for all real numbers x by

$$\mathrm{f}(x) = x^2, \qquad \mathrm{g}(x) = x - 2.$$

(i) Find the composite functions $\mathrm{fg}(x)$ and $\mathrm{gf}(x)$. [3]

(ii) Sketch the curves $y = \mathrm{f}(x)$, $y = \mathrm{fg}(x)$ and $y = \mathrm{gf}(x)$, indicating clearly which is which. [2]

3 The profit £P made by a company in its nth year is modelled by the exponential function

$$P = A\mathrm{e}^{bn}.$$

In the first year (when $n = 1$), the profit was £10 000. In the second year, the profit was £16 000.

(i) Show that $\mathrm{e}^b = 1.6$, and find b and A. [6]

(ii) What does this model predict the profit to be in the 20th year? [2]

4 When the gas in a balloon is kept at a constant temperature, the pressure P in atmospheres and the volume $V\,\mathrm{m}^3$ are related by the equation

$$P = \frac{k}{V},$$

where k is a constant. [This is known as Boyle's Law.]

When the volume is $100\,\mathrm{m}^3$, the pressure is 5 atmospheres, and the volume is increasing at a rate of $10\,\mathrm{m}^3$ per second.

(i) Show that $k = 500$. [1]

(ii) Find $\dfrac{\mathrm{d}P}{\mathrm{d}V}$ in terms of V. [2]

(iii) Find the rate at which the pressure is decreasing when $V = 100$. [4]

5 **(i)** Verify the following statement:

'$2^p - 1$ is a prime number for all prime numbers p less than 11'. [2]

(ii) Calculate 23×89, and hence disprove this statement:

'$2^p - 1$ is a prime number for all prime numbers p'. [2]

6 Fig. 6 shows the curve $e^{2y} = x^2 + y$.

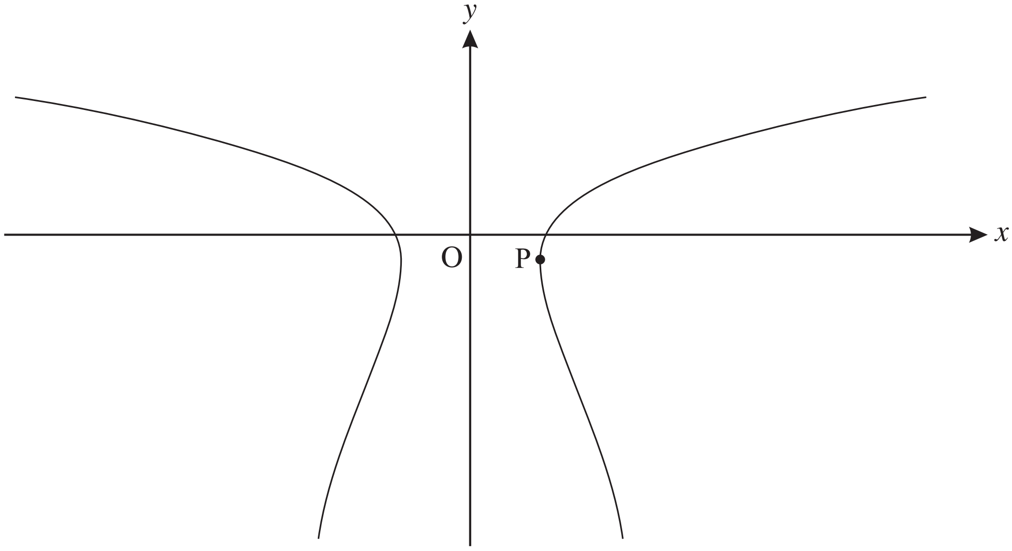


Fig. 6

(i) Show that $\frac{dy}{dx} = \frac{2x}{2e^{2y} - 1}$. [4]

(ii) Hence find to 3 significant figures the coordinates of the point P, shown in Fig. 6, where the curve has infinite gradient. [4]

Section B (36 marks)

7 A curve is defined by the equation $y = 2x \ln(1 + x)$.

(i) Find $\frac{dy}{dx}$ and hence verify that the origin is a stationary point of the curve. [4]

(ii) Find $\frac{d^2y}{dx^2}$, and use this to verify that the origin is a minimum point. [5]

(iii) Using the substitution $u = 1 + x$, show that $\int \frac{x^2}{1+x} \, dx = \int \left(u - 2 + \frac{1}{u}\right) du$.

Hence evaluate $\int_0^1 \frac{x^2}{1+x} \, dx$, giving your answer in an exact form. [6]

(iv) Using integration by parts and your answer to part **(iii)**, evaluate $\int_0^1 2x \ln(1 + x) \, dx$. [4]

8 Fig. 8 shows the curve $y = \mathrm{f}(x)$, where $\mathrm{f}(x) = 1 + \sin 2x$ for $-\frac{1}{4}\pi \leqslant x \leqslant \frac{1}{4}\pi$.

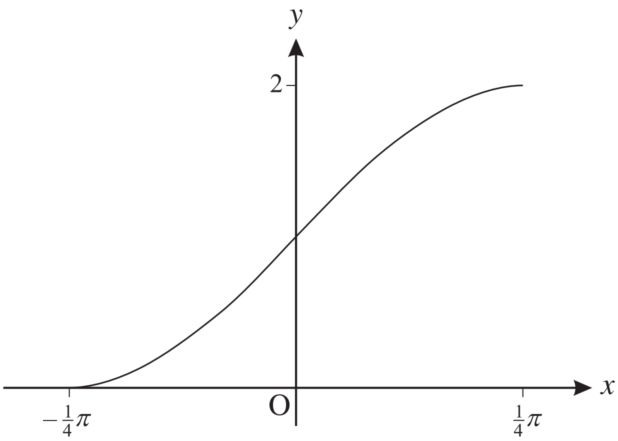


Fig. 8

(i) State a sequence of two transformations that would map part of the curve $y = \sin x$ onto the curve $y = \mathrm{f}(x)$. [4]

(ii) Find the area of the region enclosed by the curve $y = \mathrm{f}(x)$, the x-axis and the line $x = \frac{1}{4}\pi$. [4]

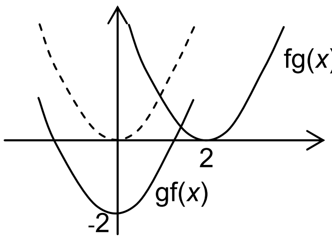
(iii) Find the gradient of the curve $y = \mathrm{f}(x)$ at the point $(0, 1)$. Hence write down the gradient of the curve $y = \mathrm{f}^{-1}(x)$ at the point $(1, 0)$. [4]

(iv) State the domain of $\mathrm{f}^{-1}(x)$. Add a sketch of $y = \mathrm{f}^{-1}(x)$ to a copy of Fig. 8. [3]

(v) Find an expression for $\mathrm{f}^{-1}(x)$. [2]

4753 (C3) Methods for Advanced Mathematics

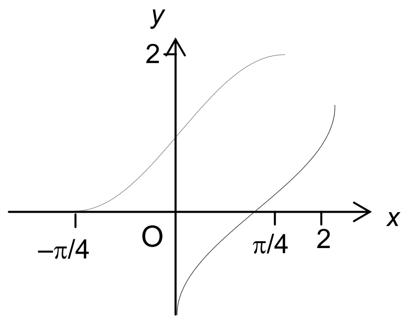
Section A

1 $y=(1+6x^2)^{1/3}$ $\Rightarrow \frac{dy}{dx}=\frac{1}{3}(1+6x^2)^{-2/3}.12x$ $=4x(1+6x^2)^{-2/3}$	M1 B1 A1 A1 [4]	chain rule used $\frac{1}{3}u^{-2/3}$ $\times 12x$ cao (must resolve 1/3 × 12) Mark final answer
2 (i) fg(x) = f(x – 2) = (x – 2)^2 gf(x) = g(x^2) = x^2 – 2.	M1 A1 A1 [3]	forming a composite function mark final answer If fg and gf the wrong way round, M1A0A0
(ii) 	B1ft B1ft [2]	fg – must have (2, 0)labelled (or inferable from scale). Condone no y-intercept, unless wrong gf – must have (0, –2) labelled (or inferable from scale) Condone no x-intercepts, unless wrong Allow ft only if fg and gf are correct but wrong way round.
3 (i) When n = 1, 10 000 = A e^b when n = 2, 16 000 = A e^{2b} $\Rightarrow \frac{16000}{10000}=\frac{Ae^{2b}}{Ae^b}=e^b$ $\Rightarrow$ e^b = 1.6 $\Rightarrow$ b = ln 1.6 = 0.470 A = 10000/1.6 = 6250.	B1 B1 M1 E1 B1 B1 [6]	soi soi eliminating A (do not allow verification) SCB2 if initial 'B's are missing, and ratio of years = 1.6 = e^b ln 1.6 or 0.47 or better (mark final answer) cao – allow recovery from inexact b's
(ii) When n = 20, P = 6250×e$^{0.470\times 20}$ = £75,550,000	M1 A1 [2]	substituting n = 20 into their equation with their A and b Allow answers from £75 000 000 to £76 000 000.
4 (i) 5 = k/100 $\Rightarrow$ k = 500*	E1 [1]	NB answer given
(ii) $\frac{dP}{dV}=-500V^{-2}=-\frac{500}{V^2}$	M1 A1 [2]	(–1)V^{-2} o.e. – allow –k/V^2
(iii) $\frac{dP}{dt}=\frac{dP}{dV}.\frac{dV}{dt}$ When V = 100, dP/dV = –500/10000 = –0.05 dV/dt = 10 $\Rightarrow$ dP/dt = –0.05 × 10 = –0.5 So P is decreasing at 0.5 Atm/s	M1 B1ft B1 A1 [4]	chain rule (any correct version) (soi) (soi) –0.5 cao

5(i) $p = 2$, $2^p - 1 = 3$, prime $p = 3$, $2^p - 1 = 7$, prime $p = 5$, $2^p - 1 = 31$, prime $p = 7$, $2^p - 1 = 127$, prime	M1 E1 [2]	Testing at least one prime testing all 4 primes (correctly) Must comment on answers being prime (allow ticks) Testing $p = 1$ is E0
(ii) $23\times 89 = 2047 = 2^{11} - 1$ 11 is prime, 2047 is not So statement is false.	M1 E1 [2]	$2^{11} - 1$ must state or imply that 11 is prime ($p = 11$ is sufficient)
6 (i) $e^{2y} = x^2 + y$ $\Rightarrow 2e^{2y}\dfrac{dy}{dx} = 2x + \dfrac{dy}{dx}$ $\Rightarrow (2e^{2y} - 1)\dfrac{dy}{dx} = 2x$ $\Rightarrow \dfrac{dy}{dx} = \dfrac{2x}{2e^{2y} - 1}$ *	M1 A1 M1 E1 [4]	Implicit differentiation – allow one slip (but with dy/dx both sides) collecting terms
(ii) Gradient is infinite when $2e^{2y} - 1 = 0$ $\Rightarrow e^{2y} = ½$ $\Rightarrow 2y = \ln ½$ $\Rightarrow y = ½ \ln ½ = -0.347$ (3 s.f.) $x^2 = e^{2y} - y = ½ - (-0.347)$ $= 0.8465$ $\Rightarrow x = 0.920$	M1 A1 M1 A1 [4]	 must be to 3 s.f. substituting their y and solving for x cao – must be to 3 s.f., but penalise accuracy once only.

Section B

7(i) $y = 2x \ln(1+x)$ $\Rightarrow \frac{dy}{dx} = \frac{2x}{1+x} + 2\ln(1+x)$ When $x = 0$, $dy/dx = 0 + 2 \ln 1 = 0$ $\Rightarrow$ origin is a stationary point.	M1 B1 A1 E1 [4]	product rule $d/dx(\ln(1+x)) = 1/(1+x)$ soi www (i.e. from correct derivative)
(ii) $\frac{d^2y}{dx^2} = \frac{(1+x).2 - 2x.1}{(1+x)^2} + \frac{2}{1+x}$ $= \frac{2}{(1+x)^2} + \frac{2}{1+x}$ When $x = 0$, $d^2y/dx^2 = 2 + 2 = 4 > 0$ $\Rightarrow$ (0, 0) is a min point	M1 A1ft A1 M1 E1 [5]	Quotient or product rule on their $2x/(1+x)$ correctly applied to their $2x/(1+x)$ o.e., e.g. $\frac{4+2x}{(1+x)^2}$ cao substituting $x = 0$ into their d^2y/dx^2 www – dep previous A1
(iii) Let $u = 1 + x \Rightarrow du = dx$ $\Rightarrow \int \frac{x^2}{1+x} dx = \int \frac{(u-1)^2}{u} du$ $= \int \frac{(u^2 - 2u + 1)}{u} du$ $= \int (u - 2 + \frac{1}{u}) du$ * $\Rightarrow \int_0^1 \frac{x^2}{1+x} dx = \int_1^2 (u - 2 + \frac{1}{u}) du$ $= \left[\frac{1}{2}u^2 - 2u + \ln u\right]_1^2$ $= 2 - 4 + \ln 2 - (\frac{1}{2} - 2 + \ln 1)$ $= \ln 2 - \frac{1}{2}$	 M1 E1 B1 B1 M1 A1 [6]	 $\frac{(u-1)^2}{u}$ www (but condone du omitted except in final answer) changing limits (or substituting back for x and using 0 and 1) $\left[\frac{1}{2}u^2 - 2u + \ln u\right]$ substituting limits (consistent with u or x) cao
(iv) $A = \int_0^1 2x \ln(1+x) dx$ Parts: $u = \ln(1+x)$, $du/dx = 1/(1+x)$ $dv/dx = 2x \Rightarrow v = x^2$ $= \left[x^2 \ln(1+x)\right]_0^1 - \int_0^1 \frac{x^2}{1+x} dx$ $= \ln 2 - \ln 2 + \frac{1}{2} = \frac{1}{2}$	 M1 A1 M1 A1 [4]	 soi substituting their $\ln 2 - \frac{1}{2}$ for $\int_0^1 \frac{x^2}{1+x} dx$ cao

8 (i) Stretch in x-direction s.f. ½ translation in y-direction 1 unit up	M1 A1 M1 A1 [4]	(in either order) – allow 'contraction' dep 'stretch' allow 'move', 'shift', etc – direction can be inferred from $\begin{pmatrix}0\\1\end{pmatrix}$ or $\begin{pmatrix}0\\1\end{pmatrix}$ dep 'translation'. $\begin{pmatrix}0\\1\end{pmatrix}$ alone is M1 A0
(ii) $A=\int_{-\pi/4}^{\pi/4}(1+\sin 2x)dx$ $=\left[x-\frac{1}{2}\cos 2x\right]_{-\pi/4}^{\pi/4}$ = π/4 – ½ cos π/2 + π/4 + ½ cos (–π/2) = π/2	M1 B1 M1 A1 [4]	correct integral and limits. Condone dx missing; limits may be implied from subsequent working. substituting their limits (if zero lower limit used, must show evidence of substitution) or 1.57 or better – cao (www)
(iii) $y = 1 + \sin 2x$ ⇒ dy/dx = 2cos 2x When x = 0, dy/dx = 2 So gradient at (0, 1) on f(x) is 2 ⇒ gradient at (1, 0) on $f^{-1}(x)$ = ½	M1 A1 A1ft B1ft [4]	differentiating – allow 1 error (but not x + 2cos 2x) If 1, then must show evidence of using reciprocal, e.g. 1/1
(iv) Domain is $0 \le x \le 2$. 	B1 M1 A1 [3]	Allow 0 to 2, but not $0 < x < 2$ or y instead of x clear attempt to reflect in $y = x$ correct domain indicated (0 to 2), and reasonable shape
(v) $y = 1 + \sin 2x$ $x \leftrightarrow y$ $x = 1 + \sin 2y$ ⇒ $\sin 2y = x – 1$ ⇒ $2y = \arcsin(x – 1)$ ⇒ $y = ½ \arcsin(x – 1)$	M1 A1 [2]	or $\sin 2x = y – 1$ cao

4753: Methods for Advanced Mathematics (C3) (Written Examination)

General Comments

The paper proved to be a fair test of students' ability. There were plenty of accessible marks, and even weak candidates managed to score around 20 marks. There were also many excellent scripts over 60 marks. The scripts suggested signs of improvement on some topics which have caused problems in the past, such as implicit differentiation and inverse trigonometric functions. Although virtually all candidates attempted all the questions, there were signs in question 8 of a few candidates running out of time. The standard of presentation was variable.

Algebraic immaturity is a common source of problems. For example, the inability to eliminate variables from pairs of simultaneous equations in question 3 and errors in simplification of expressions in questions 1 and 7(ii), were common sources of weakness. Another general point is candidates' sensitivity to command words such as 'show' and 'verify', for example using 'verification' in question 3(i) and 'showing' in 7(i).

Comments on Individual Questions

Section A

1) This was a straightforward test of the chain rule, in which most competent candidates scored full marks. Sources of error were using the wrong index for the cube root, derivative formula errors, and faulty simplification of the final result.

2) This question was well done, with nearly all candidates applying f and g in the correct order for each composite function, and sketching the resulting quadratic functions. In the sketch, only the *x*-intercept for fg and *y*-intercept for gf were required, though some 'burned their boats' by getting the other intercept wrong.

3) Weaker candidates made heavy weather of eliminating A and establishing $e^b = 1.6$ – many substituted the given result prematurely. However, nearly all candidates scored B1 for $A = 6250$ and $b = 0.470$, and the 2 easy marks for part (ii). The answers to part (ii) were often written to improbable degrees of accuracy, though this was not penalised.

4) This question was less successfully answered. Part (i) was a gift of a mark for everyone, but the easy derivative in part (ii) was disappointingly done, the main errors being $(v-k)/v^2$ and k lnv. The use of the chain rule in part (iii) was less secure, with many candidates thinking that the rate of change of P was dP/dV.

5) This little 4-mark question rarely achieved full marks. In part (i), we penalised those who tested $p = 1$ – clearly many candidates classify 1 as a prime. In part (ii), although most candidates showed that $2^{11} - 1 = 2047$ for M1, many failed to complete the proof convincingly by pointing out that 11 is prime, and 2047 is not.

6) The implicit differentiation in part (i) was usually either 0 marks or 4 marks, with many candidates scoring full marks. Part (ii) was less well done, with numerator = 0, or numerator = denominator, common errors. A fair number of candidates lost an accuracy mark by writing $x = 0.92$ instead of 0.920.

Section B

7) Most candidates scored over half marks for this question, but a lot of marks were lost if they failed to find the correct derivative of $\ln(1 + x)$.

(i) The product rule was well known, though the derivative of ln(1 + x) was occasionally wrong – 1/x a common error. It is important that candidates understand the meaning of command words such as 'verify': having found dy/dx, a significant number of students tried to solve dy/dx = 0 instead of verifying that x=0 is a solution.

(ii) Some candidates lost their way with the second derivative by combining the fractions instead of differentiating them separately, then using the quotient rule on the more complicated expression. The quotient rule was generally well done. The second derivative rule was also well known, though the final 'E' mark was reserved for better candidates who had managed the algebra securely.

(iii) This was a fairly easy integration by substitution, with the result given. However, candidates needed to show $du = dx$, and to expand $(u - 1)^2$, to gain the 'E' mark. Integration errors were usually from the $1/u$ term, and some candidates got the limits for u and x mixed up.

(iv) There were plenty of correct applications of integration by parts, but occasionally flawed notation led to the substitution of the answer from part (iii) incorrectly – for example, it was included inside the brackets with the limits resulting in it being added and subtracted.

8) This question tested a variety of topics, and rarely gained full marks. Some attempts showed evidence of rushing through lack of time.

(i) To gain full marks, we wanted the transformations described correctly as 'translation' and 'stretch', and the directions clearly indicated. Getting the directions mixed up was quite a common error.

(ii) A surprising number of candidates, even competent ones, thought that the limits of this integral were from 0 (rather than $-\pi/4$) to $\pi/4$, and integration errors were quite common.

(iii) The first three marks were very straightforward, though mixing derivative and integral results for trigonometric functions is quite common. The derivative of the inverse function as the reciprocal was often not known – some candidates tried to derive the inverse function, and others confused this with the perpendicular gradient result, giving the answer as − ½ instead of ½ .

(iv) Domains and ranges often cause problems, and correct domains here were relatively rare. Most candidates attempted a reflection in $y = x$ and gained M1, but few showed the correct domain for the inverse function for the A1.

(v) Many candidates got these two marks, and even weaker ones usually got M1 for attempts to invert. The most common error was confusing the inverse with the reciprocal of the function.