OCR
RECOGNISING ACHIEVEMENT

ADVANCED GCE **4753/01**

MATHEMATICS (MEI)

Methods for Advanced Mathematics (C3)

MONDAY 2 JUNE 2008 Morning

Time: 1 hour 30 minutes

Additional materials (enclosed): None

Additional materials (required):
Answer Booklet (8 pages)
Graph paper
MEI Examination Formulae and Tables (MF2)

INSTRUCTIONS TO CANDIDATES

- Write your name in capital letters, your Centre Number and Candidate Number in the spaces provided on the Answer Booklet.
- Read each question carefully and make sure you know what you have to do before starting your answer.
- Answer **all** the questions.
- You are permitted to use a graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets [ ] at the end of each question or part question.
- The total number of marks for this paper is **72**.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.

This document consists of **4** printed pages.

[Turn over

Section A (36 marks)

1 Solve the inequality $|2x - 1| \leqslant 3$. [4]

2 Find $\int x e^{3x}\,dx$. [4]

3 **(i)** State the algebraic condition for the function $f(x)$ to be an even function.

What geometrical property does the graph of an even function have? [2]

(ii) State whether the following functions are odd, even or neither.

(A) $f(x) = x^2 - 3$

(B) $g(x) = \sin x + \cos x$

(C) $h(x) = \dfrac{1}{x + x^3}$ [3]

4 Show that $\displaystyle\int_1^4 \frac{x}{x^2 + 2}\,dx = \tfrac{1}{2}\ln 6$. [4]

5 Show that the curve $y = x^2 \ln x$ has a stationary point when $x = \dfrac{1}{\sqrt{e}}$. [6]

6 In a chemical reaction, the mass m grams of a chemical after t minutes is modelled by the equation

$$m = 20 + 30e^{-0.1t}.$$

(i) Find the initial mass of the chemical.

What is the mass of chemical in the long term? [3]

(ii) Find the time when the mass is 30 grams. [3]

(iii) Sketch the graph of m against t. [2]

7 Given that $x^2 + xy + y^2 = 12$, find $\dfrac{dy}{dx}$ in terms of x and y. [5]

Section B (36 marks)

8 Fig. 8 shows the curve $y = \mathrm{f}(x)$, where $\mathrm{f}(x) = \dfrac{1}{1 + \cos x}$, for $0 \leqslant x \leqslant \frac{1}{2}\pi$.

P is the point on the curve with x-coordinate $\frac{1}{3}\pi$.

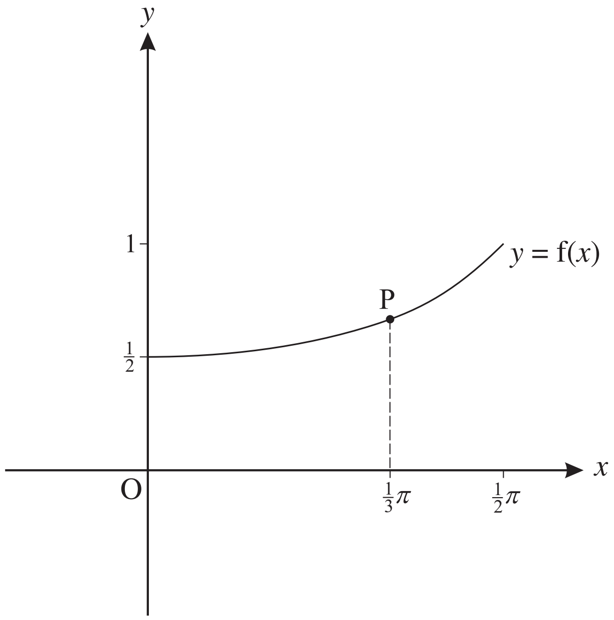


Fig. 8

(i) Find the y-coordinate of P. [1]

(ii) Find $\mathrm{f}'(x)$. Hence find the gradient of the curve at the point P. [5]

(iii) Show that the derivative of $\dfrac{\sin x}{1 + \cos x}$ is $\dfrac{1}{1 + \cos x}$. Hence find the exact area of the region enclosed by the curve $y = \mathrm{f}(x)$, the x-axis, the y-axis and the line $x = \frac{1}{3}\pi$. [7]

(iv) Show that $\mathrm{f}^{-1}(x) = \arccos\left(\dfrac{1}{x} - 1\right)$. State the domain of this inverse function, and add a sketch of $y = \mathrm{f}^{-1}(x)$ to a copy of Fig. 8. [5]

[Question 9 is printed overleaf.]

9 The function $f(x)$ is defined by $f(x) = \sqrt{4 - x^2}$ for $-2 \leqslant x \leqslant 2$.

(i) Show that the curve $y = \sqrt{4 - x^2}$ is a semicircle of radius 2, and explain why it is not the whole of this circle. [3]

Fig. 9 shows a point P (a, b) on the semicircle. The tangent at P is shown.

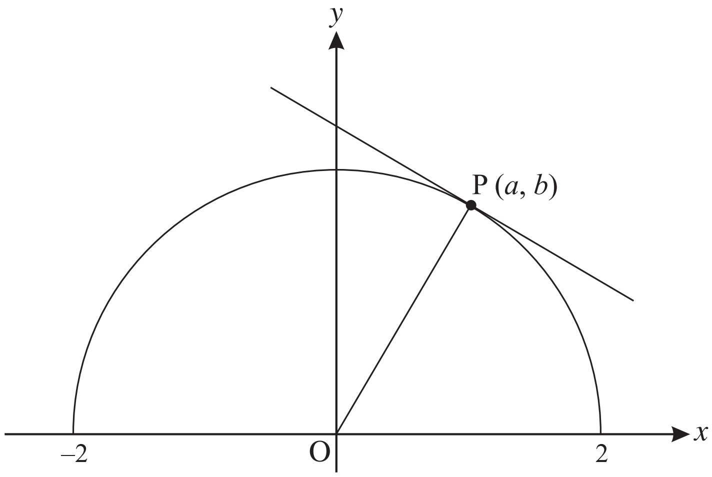


Fig. 9

(ii) (A) Use the gradient of OP to find the gradient of the tangent at P in terms of a and b.

(B) Differentiate $\sqrt{4 - x^2}$ and deduce the value of $f'(a)$.

(C) Show that your answers to parts (A) and (B) are equivalent. [6]

The function $g(x)$ is defined by $g(x) = 3f(x - 2)$, for $0 \leqslant x \leqslant 4$.

(iii) Describe a sequence of two transformations that would map the curve $y = f(x)$ onto the curve $y = g(x)$.

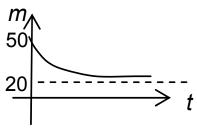
Hence sketch the curve $y = g(x)$. [6]

(iv) Show that if $y = g(x)$ then $9x^2 + y^2 = 36x$. [3]

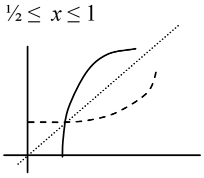
4753 (C3) Methods for Advanced Mathematics

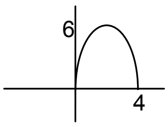
Section A

1 $\|2x-1\|\le 3$ $\Rightarrow$ $-3 \le 2x-1 \le 3$ $\Rightarrow$ $-2 \le 2x \le 4$ $\Rightarrow$ $-1 \le x \le 2$ *or* $(2x-1)^2 \le 9$ $\Rightarrow$ $4x^2-4x-8 \le 0$ $\Rightarrow$ $(4)(x+1)(x-2) \le 0$ $\Rightarrow$ $-1 \le x \le 2$	M1 A1 M1 A1 M1 A1 A1 A1 [4]	$2x-1 \le 3$ (or =) $x \le 2$ $2x-1 \ge -3$ (or =) $x \ge -1$ squaring and forming quadratic = 0 (or $\le$) factorising or solving to get $x = -1, 2$ $x \ge -1$ $x \le 2$ (www)
2 Let $u = x$, $dv/dx = e^{3x} \Rightarrow v = e^{3x}/3$ $\Rightarrow$ $\int x e^{3x}\,dx = \frac{1}{3}xe^{3x} - \int \frac{1}{3}e^{3x}.1.dx$ $= \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + c$	M1 A1 A1 B1 [4]	parts with $u = x$, $dv/dx = e^{3x} \Rightarrow v$ $= \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x}$ $+c$
3 (i) $f(-x) = f(x)$ Symmetrical about Oy.	B1 B1 [2]	
(ii) (A) even (B) neither (C) odd	B1 B1 B1 [3]	
4 Let $u = x^2 + 2 \Rightarrow du = 2x\,dx$ $\int_1^4 \frac{x}{x^2+2}dx = \int_3^{18} \frac{1/2}{u}du$ $= \frac{1}{2}[\ln u]_3^{18}$ $= ½ (\ln 18 - \ln 3)$ $= ½ \ln(18/3)$ $= ½ \ln 6$*	M1 A1 M1 E1 [4]	$\int \frac{1/2}{u}du$ or $k \ln (x^2 + 1)$ ½ ln u or ½ ln(x^2 + 2) substituting correct limits (u or x) must show working for ln 6
5 $y = x^2 \ln x$ $\Rightarrow$ $\frac{dy}{dx} = x^2.\frac{1}{x} + 2x\ln x$ $= x + 2x\ln x$ $dy/dx = 0$ when $x + 2x\ln x = 0$ $\Rightarrow$ $x(1 + 2\ln x) = 0$ $\Rightarrow$ $\ln x = -½$ $\Rightarrow$ $x = e^{-½} = 1/\sqrt{e}$ *	M1 B1 A1 M1 M1 E1 [6]	product rule $d/dx (\ln x) = 1/x$ soi oe their deriv = 0 or attempt to verify $\ln x = -½ \Rightarrow x = e^{-½}$ or $\ln (1/\sqrt{e}) = -½$

6(i) Initial mass = 20 + 30 e^0 = 50 grams Long term mass = 20 grams	M1A1 B1 [3]	
(ii) 30 = 20 + 30 $e^{-0.1t}$ $\Rightarrow$ $e^{-0.1t}$ = 1/3 $\Rightarrow$ $-0.1t = \ln(1/3) = -1.0986\ldots$ $\Rightarrow$ t = 11.0 mins	M1 M1 A1 [3]	 anti-logging correctly 11, 11.0, 10.99, 10.986 (not more than 3 d.p)
(iii) 	B1 B1 [2]	correct shape through (0, 50) – ignore negative values of t $\rightarrow$ 20 as $t \rightarrow \infty$
7 $x^2 + xy + y^2 = 12$ $\Rightarrow$ $2x + x\frac{dy}{dx} + y + 2y\frac{dy}{dx} = 0$ $\Rightarrow$ $(x+2y)\frac{dy}{dx} = -2x - y$ $\Rightarrow$ $\frac{dy}{dx} = -\frac{2x+y}{(x+2y)}$	M1 B1 A1 M1 A1 [5]	Implicit differentiation $x\frac{dy}{dx} + y$ correct equation collecting terms in dy/dx and factorising oe cao

Section B

8(i) $y = 1/(1+\cos\pi/3) = 2/3$.	B1 [1]	or 0.67 or better
(ii) $f'(x) = -1(1+\cos x)^{-2}.-\sin x$ $= \dfrac{\sin x}{(1+\cos x)^2}$ When $x = \pi/3$, $f'(x) = \dfrac{\sin(\pi/3)}{(1+\cos(\pi/3))^2}$ $= \dfrac{\sqrt{3}/2}{(1\frac{1}{2})^2} = \dfrac{\sqrt{3}}{2}\times\dfrac{4}{9} = \dfrac{2\sqrt{3}}{9}$	M1 B1 A1 M1 A1 [5]	chain rule or quotient rule d/dx (cos x) = –sin x soi correct expression substituting $x = \pi/3$ oe or 0.38 or better. (0.385, 0.3849)
(iii) deriv = $\dfrac{(1+\cos x)\cos x - \sin x.(-\sin x)}{(1+\cos x)^2}$ $= \dfrac{\cos x + \cos^2 x + \sin^2 x}{(1+\cos x)^2}$ $= \dfrac{\cos x + 1}{(1+\cos x)^2}$ $= \dfrac{1}{1+\cos x}$ * Area $= \int_0^{\pi/3} \dfrac{1}{1+\cos x}dx$ $= \left[\dfrac{\sin x}{1+\cos x}\right]_0^{\pi/3}$ $= \dfrac{\sin\pi/3}{1+\cos\pi/3}(-0)$ $= \dfrac{\sqrt{3}}{2}\times\dfrac{2}{3} = \dfrac{\sqrt{3}}{3}$	M1 A1 M1dep E1 B1 M1 A1 cao [7]	Quotient or product rule – condone uv' – u'v for M1 correct expression $\cos^2 x + \sin^2 x = 1$ used dep M1 www substituting limits or $1/\sqrt{3}$ - must be exact
(iv) $y = 1/(1 + \cos x)$ $x \leftrightarrow y$ $x = 1/(1 + \cos y)$ $\Rightarrow$ $1 + \cos y = 1/x$ $\Rightarrow$ $\cos y = 1/x - 1$ $\Rightarrow$ $y = \arccos(1/x - 1)$ * Domain is $\frac{1}{2} \le x \le 1$ 	M1 A1 E1 B1 B1 [5]	attempt to invert equation www reasonable reflection in $y = x$

9 (i) $y = \sqrt{4-x^2}$ $\Rightarrow$ $y^2 = 4 - x^2$ $\Rightarrow$ $x^2 + y^2 = 4$ which is equation of a circle centre O radius 2 Square root does not give negative values, so this is only a semi-circle.	M1 A1 B1 [3]	squaring $x^2 + y^2 = 4$ + comment (correct) oe, e.g. f is a function and therefore single valued
(ii) (*A*) Grad of OP = b/a $\Rightarrow$ grad of tangent = $-\frac{a}{b}$ (*B*) $f'(x) = \frac{1}{2}(4-x^2)^{-1/2}.(-2x)$ $= -\frac{x}{\sqrt{4-x^2}}$ $\Rightarrow$ $f'(a) = -\frac{a}{\sqrt{4-a^2}}$ (*C*) $b = \sqrt{(4 - a^2)}$ so $f'(a) = -\frac{a}{b}$ as before	M1 A1 M1 A1 B1 E1 [6]	 chain rule or implicit differentiation oe substituting a into their $f'(x)$
(iii) Translation through $\begin{pmatrix}2\\0\end{pmatrix}$ followed by stretch scale factor 3 in y-direction 	M1 A1 M1 A1 M1 A1 [6]	Translation in x-direction through $\begin{pmatrix}2\\0\end{pmatrix}$ or 2 to right ('shift', 'move' M1 A0) $\begin{pmatrix}2\\0\end{pmatrix}$ alone is SC1 stretch in y-direction (condone y 'axis') (scale) factor 3 elliptical (or circular) shape through (0, 0) and (4, 0) and (2, 6) (soi) –1 if whole ellipse shown
(iv) $y = 3f(x - 2)$ $= 3\sqrt{(4 - (x - 2)^2)}$ $= 3\sqrt{(4 - x^2 + 4x - 4)}$ $= 3\sqrt{(4x - x^2)}$ $\Rightarrow$ $y^2 = 9(4x - x^2)$ $\Rightarrow$ $9x^2 + y^2 = 36x$ *	M1 A1 E1 [3]	or substituting $3\sqrt{(4 - (x - 2)^2)}$ oe for y in $9x^2 + y^2$ $4x - x^2$ www

4753 Methods for Advanced Mathematics (C3) (Written Examination)

General Comments

This paper proved to be a fair and accessible test, and many candidates achieved over 50 marks. Section A was often very well done – well prepared candidates were able to tackle these questions confidently and lost few marks. Question 8 of section B also scored well, but the final question tested all candidates and discriminated quite well. All but few also had ample time to complete all the questions.

In general, there is a considerable amount of calculus tested by this paper, and students are usually well prepared in calculus techniques such as chain, product and quotient rules, and integration by parts and substitution. There is some confusion between differentials and integrals of trigonometric, logarithmic and exponential functions. The language of functions, and work on the modulus function, are perhaps less secure.

The paper proved to be quite forgiving about notational weaknesses, and there were quite a few generous 'E' marks. Any candidate scoring less than 20 marks on this paper should arguably not have been entered, and it is pleasing to report that this occurred relatively infrequently. However, algebraic fragility continues to mar the work of candidates, although we often, with reluctance, use the principle of ignoring subsequent working ('isw') in marking correct formulations of answers which are then corrupted by poor algebra. The number of candidates who, for example, leave expressions such as x^2/x unsimplified, or omit essential brackets, or write $(1 + \cos x)^2 = 1 + \cos^2 x$, is disappointing!

Several questions involved sketches of graphs – these are far better done *without* graph paper – it is indeed extremely unlikely that graph paper is required to be used in this paper.

Comments on Individual Questions

Section A

1) Most candidates correctly derived $x \le 2$ and scored 2 marks out of 4, but many had difficulties in handling the left hand boundary of the inequality, for example arguing that $-2x \le 2 \Rightarrow x \le -1$. Writing $|2x| \le 4$ was penalised. Some candidates squared both sides, but some of these made the error of arguing that $(x + 1)(x - 2) \le 1 \Rightarrow x \le -1$ and $x \le 2$. The best solutions worked with each side simultaneously. Very few students used a graphical approach.

2) Although there were many completely correct solutions, the constant of integration was frequently missing (though this equally gave an easy mark to some candidates who made a mess of the integration). Those who failed to do the integration correctly often confused integration by parts with the product rule, or integrated e^{3x} as $3e^{3x}$.

3) The algebraic and geometric conditions for an even function were generally well known, though some candidates omitted the axis of symmetry from the latter. Most classified at least one of f, g or h correctly (albeit perhaps fortuitously); the function g giving the most problems – many thought this was 'odd'.

4) Many candidates scored full marks for this question, with a pleasing number using inspection rather than substitution. We needed to see the step $\ln 18 - \ln 3 = \ln(18/3)$ which was occasionally incorrectly written $\ln 18 / \ln 3$. Although sloppy working (for example, omitting 'dx' or 'du', limits inconsistent with these, etc.) was generally condoned here, this has been penalised in some recent papers, and it would be pleasing to see less of this.

5) The product rule and the derivative of $\ln x$ were well known, and most candidates found the derivative correctly. However, quite a few failed to score the last two marks – it is important with a given answer that enough working is shown, and they needed to show the step from $\ln x = -½$ to $x = e^{-½}$, or, if verifying but substituting $x = e^{-½}$ into the derivative, that $\ln(e^{-½}) = -½$. The latter method was quite common, and accepted. A number of candidates left x^2/x unsimplified – as the derivative had not been asked for explicitly, this was not penalised.

6) This exponential decay question proved to be a 'banker' for even the weakest candidates. The first three marks were almost always awarded, and solving the exponential equation in part (ii) using logarithms is very well done. Some candidates are still plotting graphs – we much prefer to see a sketch *not* on graph paper, and this in effect makes the asymptotic behaviour *easier* to credit than a graph which 'stops' at a plotted point, especially if the asymptote $m = 20$ is shown. Occasionally the vertical axis was erroneously marked as an asymptote.

7) This provided the most challenging test in section A. In particular, differentiating xy using the product rule was achieved by better candidates only. The error of starting an implicit differentiation by stating dy/dx = … is still quite common, and sometimes compounded by some candidates then proceeding to collect the extraneous dy/dx term in with the other terms in dy/dx! Another common error was to 'forget' to equate the derivative of the constant 12 to zero.

Section B

8) This question was in general done well, with most candidates gaining well over half marks. There is, however, quite a lot of confusion between differentials and integrals of $\sin x$ and $\cos x$, perhaps encouraged by the reasoning required in part (iii).

(i) This was an easy mark for all and sundry. However, some weaker candidates used calculators in degree mode when substituting $x = \pi/3$.

(ii) Quite a few candidates used a quotient rule rather than a chain rule, and omitted the 'zero' derivative du/dx. Sign errors were common, as were omitted brackets.

(iii) Most candidates attempted a quotient rule, but again this was often marred by poor algebra, for example sign errors, incorrect formulae, $(1 + \cos x)^2 = 1 + \cos^2 x$, and indiscriminate cancelling. However, competent candidates often achieved full marks. In the second part, weaker candidates missed the point, and often started integrating $\sin x / (1 + \cos x)$ as $-\ln (1 + \cos x)$. Some also failed to express the final answer exactly.

(iv) Most candidates managed to invert the function correctly, but very few gained the mark for the correct domain. The sketches were often well done – most candidates attempted a reasonable reflection in $y = x$, and we condoned domain and range errors in this instance.

9) This question was less successfully answered than question 8, especially the final part. The language of functions is a topic which requires sound notation and conceptual understanding, and this often found out less secure candidates.

(i) Most candidates realised that this was about squaring and comparing with the circle equation, and some did it in reverse by square rooting. However, explanations about why the full circle was not represented by the function were often unconvincing, mixing up the ideas ‘square roots cannot be negative’ and ‘you cannot square root negative numbers’. We also allowed well expressed arguments about functions being single valued.

(ii) More assured candidates followed the three steps correctly, with correct notation. However, many missed the point, failing at the first hurdle to write down the gradient of OP as b/a. Many, although spotting a chain rule, missed the negative sign in differentiating y, and failed to substitute $x = a$ in their derivative to give $f'(a)$.

(iii) The stretch and the translation can of course be done in either order. We wanted to see ‘stretch’ and ‘translation’, and penalised other descriptions such as ‘move’, ‘shift’, ‘multiply the *y*-coordinate by 3’, which were common. Sketches of the curve which went through (0, 0), (4, 0) and (2, 6) usually gained full marks, notwithstanding weaknesses in their elliptical shape.

(iv) Success in this part proved to be the preserve of more assured candidates. The lynch pin was to write $y = 3\sqrt{(4 - (x - 2)^2)}$ – without this, little progress was made. Expanding the bracket and handling the negatives then caused some candidates to lose their way.