OCR
RECOGNISING ACHIEVEMENT

ADVANCED GCE

MATHEMATICS (MEI) **4758/01**

Differential Equations

Candidates answer on the Answer Booklet

OCR Supplied Materials:
- 8 page Answer Booklet
- Graph paper
- MEI Examination Formulae and Tables (MF2)

Other Materials Required:
None

Wednesday 20 May 2009
Afternoon

Duration: 1 hour 30 minutes

INSTRUCTIONS TO CANDIDATES

- Write your name clearly in capital letters, your Centre Number and Candidate Number in the spaces provided on the Answer Booklet.
- Use black ink. Pencil may be used for graphs and diagrams only.
- Read each question carefully and make sure that you know what you have to do before starting your answer.
- Answer any **three** questions.
- Do **not** write in the bar codes.
- You are permitted to use a graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.
- The acceleration due to gravity is denoted by $g\,\text{m s}^{-2}$. Unless otherwise instructed, when a numerical value is needed, use $g = 9.8$.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets **[ ]** at the end of each question or part question.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.
- The total number of marks for this paper is **72**.
- This document consists of **4** pages. Any blank pages are indicated.

Turn over

1 A car travels over a rough surface. The vertical motion of the front suspension is modelled by the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 25y = 20\cos 5t,$$

where y is the vertical displacement of the top of the suspension and t is time.

(i) Find the general solution. **[8]**

Initially $y = 1$ and $\frac{\mathrm{d}y}{\mathrm{d}t} = 0$.

(ii) Find the solution subject to these conditions. **[4]**

(iii) Sketch the solution curve for $t \geqslant 0$. **[4]**

A refined model of the motion of the suspension is given by

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 2\frac{\mathrm{d}y}{\mathrm{d}t} + 25y = 20\cos 5t.$$

(iv) Verify that $y = 2\sin 5t$ is a particular integral for this differential equation. Hence find the general solution. **[6]**

(v) Compare the behaviour of the suspension predicted by the two models. **[2]**

2 The differential equation

$$x\frac{\mathrm{d}y}{\mathrm{d}x} + 3y = \frac{\sin x}{x}$$

is to be solved for $x > 0$.

(i) Find the general solution for y in terms of x. **[9]**

As $x \to 0$, y tends to a finite limit.

(ii) Use the approximations $\sin x \approx x - \frac{1}{6}x^3$ and $\cos x \approx 1 - \frac{1}{2}x^2$ (both valid for small x) to find the value of the arbitrary constant and the limiting value of y as $x \to 0$. Hence state the particular solution. **[6]**

(iii) Show that, when $y = 0$, $\tan x = x$. **[2]**

An alternative method of investigating the behaviour of y for small x is to use the approximation $\sin x \approx x - \frac{1}{6}x^3$ in the differential equation, giving

$$x\frac{\mathrm{d}y}{\mathrm{d}x} + 3y = \frac{x - \frac{1}{6}x^3}{x}.$$

(iv) Solve this differential equation and, given that y tends to a finite limit as $x \to 0$, show that the value of the limit is the same as that found in part **(ii)**. **[7]**

3 **(a)** An electric circuit has an inductor and a resistor in series with an alternating power source. The circuit is switched on and after t seconds the current is I amps. The current satisfies the differential equation

$$2\frac{\mathrm{d}I}{\mathrm{d}t} + 4I = 3\cos 2t.$$

(i) Find the complementary function and a particular integral. Hence state the general solution for I in terms of t. **[8]**

Initially the current is zero.

(ii) Find the particular solution. **[2]**

(iii) Calculate the amplitude of the current for large values of t. Sketch the solution curve for large values of t. **[4]**

(b) The displacement, y, of a particle at time t satisfies the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 2 - 2y + \mathrm{e}^{-t}.$$

You are **not** required to solve this differential equation.

The particle initially has displacement zero. The displacement has only one stationary value, which is where $y = \frac{9}{8}$. Also the velocity of the particle tends to zero as $t \to \infty$.

(i) Without solving the differential equation, use it to find

(A) the gradient of the solution curve when $t = 0$; **[2]**

(B) the value of t at the stationary value of y; **[3]**

(C) the limit of y as $t \to \infty$. **[2]**

(ii) Hence sketch the solution curve for $t \geqslant 0$, illustrating these results. **[3]**

4 The simultaneous differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 7x + 6y + 2\mathrm{e}^{-3t}$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = -12x - 10y + 5\sin t$$

are to be solved for $t \geqslant 0$.

(i) Show that

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 3\frac{\mathrm{d}x}{\mathrm{d}t} + 2x = 14\mathrm{e}^{-3t} + 30\sin t.$$ **[5]**

(ii) Show that this differential equation has a particular integral of the form $x = a\mathrm{e}^{-3t} - 9\cos t + 3\sin t$, where a is a constant to be determined.

Hence find the general solution for x in terms of t. **[8]**

(iii) Find the corresponding general solution for y. **[4]**

(iv) Show that, for large values of t, $x = y$ when $\tan t \approx k$, where k is a constant to be determined. **[4]**

(v) Find the ratio of the amplitudes of y and x for large values of t. **[3]**

4758 Differential Equations

1(i)	$\alpha^2+25=0$	M1		Auxiliary equation
	$\alpha=\pm 5j$	A1		
	CF $y=A\cos 5t+B\sin 5t$	F1		CF for their roots
	PI $y=at\cos 5t+bt\sin 5t$	B1		
	$\dot{y}=a\cos 5t-5at\sin 5t+b\sin 5t+5bt\cos 5t$			
	$\ddot{y}=-10a\sin 5t-25at\cos 5t+10b\cos 5t-25bt\sin 5t$	M1		Differentiate twice
	In DE $\Rightarrow 10b\cos 5t-10a\sin 5t=20\cos 5t$	M1		Substitute and compare coefficients
	$\Rightarrow b=2, a=0$	A1		
	PI $y=2t\sin 5t$			
	GS $y=2t\sin 5t+A\cos 5t+B\sin 5t$	F1	**8**	
(ii)	$t=0, y=1\Rightarrow A=1$	B1		From correct GS
	$\dot{y}=2\sin 5t+10t\cos 5t-5A\sin 5t+5B\cos 5t$	M1		Differentiate
	$t=0, \dot{y}=0\Rightarrow B=0$	M1		Use condition on $\dot{y}$
	$y=2t\sin 5t+\cos 5t$	A1	**4**	
(iii)	Curve through (0,1)	B1		
	Curve with zero gradient at (0,1)	B1		
	Oscillations	B1		
	Oscillations with increasing amplitude	B1	**4**	
(iv)	$y=2\sin 5t, \dot{y}=10\cos 5t, \ddot{y}=-50\sin 5t$			
	$\ddot{y}+2\dot{y}+25y=-50\sin 5t+20\cos 5t+50\sin 5t$	M1		Substitute into DE
	$=20\cos 5t$	E1		
	$\alpha^2+2\alpha+25=0$	M1		Auxiliary equation
	$\alpha=-1\pm\sqrt{24}j$	A1		
	CF $e^{-t}(C\cos\sqrt{24}t+D\sin\sqrt{24}t)$	F1		CF for their complex roots
	GS $y=2\sin 5t+e^{-t}(C\cos\sqrt{24}t+D\sin\sqrt{24}t)$	F1	**6**	Their PI + their CF with two arbitrary constants
(v)	Oscillations of amplitude 2	B1		or bounded oscillations; or both oscillate
	Compared to unbounded oscillations in first model	B1	**2**	or equivalent; or one bounded, one unbounded

2(i)	$\frac{dy}{dx}+\frac{3}{x}y=\frac{\sin x}{x^2}$	M1	Rearrange
	$I = \exp\int\frac{3}{x}\,dx$	M1	Attempting integrating factor
	$= \exp(3\ln x)$	A1	
	$= x^3$	A1	Correct and simplified
	$\frac{d}{dx}\left(x^3y\right) = x\sin x$	M1	Multiply and recognise derivative
	$x^3y=\int x\sin x\,dx=-x\cos x+\int\cos x\,dx$	M1	Integrate
		A1	
	$=-\cos x+\sin x+A$	A1	All correct
	$y=(-x\cos x+\sin x+A)/\,x^3$	F1	Must include constant
		9	
(ii)	$y\approx\left(-x\left(1-\frac{1}{2}x^2\right)+x-\frac{1}{6}x^3+A\right)/\,x^3$	M1	Substitute given approximations
		F1	
	$=\frac{1}{3}+\frac{A}{x^3}$	M1	Use finite limit to deduce A
	$A = 0$	A1	
	$y=(\sin x=x\cos x)/\,x^3$	B1	Correct particular solution
	$\lim_{x\to 0}y=\frac{1}{3}$	B1	Correct limit
		6	
(iii)	$y=0\Rightarrow\sin x-x\cos x=0$	M1	Equate to zero and attempt to get tan*x*
	$\Rightarrow\tan x=x$	E1	Convincingly shown
		2	
(iv)	$\frac{dy}{dx}+\frac{3}{x}y=\frac{1}{x}-\frac{1}{6}x$, multiply by $I=x^3$	M1	Rearrange and multiply by IF
		B1	Same IF as in (i) or correct IF
	$\frac{d}{dx}(x^3y)=x^2-\frac{1}{6}x^4$	A1	Recognise derivative and RHS correct
	$x^3y=\frac{1}{3}x^3-\frac{1}{30}x^5+B$	M1	Integrate
	$y=\frac{1}{3}-\frac{1}{30}x^2+\frac{B}{x^3}$	A1	cao
	Finite limit $\Rightarrow B = 0$	M1	Use condition to find constant

	$\lim_{x\to 0} y = \frac{1}{3}$	E1	Show correct limit (or same limit as (ii))
		7	
3(a)(i)	$2\alpha + 4 = 0 \Rightarrow \alpha = -2$	M1	Find root of auxiliary equation
	CF Ae^{-2t}	A1	
	PI $I = a\cos 2t + b\sin 2t$	B1	
	$\dot{I} = -2a\sin 2t + 2b\cos 2t$	M1	Differentiate
	$-4a\sin 2t + 4b\cos 2t + 4a\cos 2t + 4b\sin 2t = 3\cos 2t$	M1	Substitute
	$-4a + 4b = 0, 4b + 4a = 3 \Rightarrow a = b = \frac{3}{8}$	M1	Compare coefficients and solve
	PI $I = \frac{3}{8}(\cos 2t + \sin 2t)$	A1	
	GS $I = Ae^{-2t} + \frac{3}{8}(\cos 2t + \sin 2t)$	F1 **8**	Their PI + their CF with *one* arbitrary constant
(ii)	$t = 0, I = 0 \Rightarrow 0 = A + \frac{3}{8} \Rightarrow A = -\frac{3}{8}$	M1	Use condition
	$I = \frac{3}{8}(\cos 2t + \sin 2t - e^{-2t})$	A1 **2**	cao
(iii)	For large t, $I \approx \frac{3}{8}(\cos 2t + \sin 2t)$	M1	Consider behaviour for large t (may be implied)
	Amplitude = $\frac{3}{8}\sqrt{1^2 + 1^2} = \frac{3}{8}\sqrt{2}$	A1	
	Curve with oscillations with constant amplitude	B1	
	Their amplitude clearly indicated	B1 **4**	
(b)(i)	(*A*) $t = 0$, $y = 0 \Rightarrow \frac{dy}{dt} = 2 - 2(0) + e^0$	M1	Substitute into DE
	Gradient =3	A1	
	(*B*) At stationary point, $\frac{dy}{dt} = 0, y = \frac{9}{8}$	M1	Substitute into DE
	$\Rightarrow 0 = 2 - 2(\frac{9}{8}) + e^{-t} \Rightarrow e^{-t} = \frac{1}{4}$	M1	Solve for t
	$\Rightarrow t =$ ln4	A1	
	(*C*) $\frac{dy}{dt} \to 0, e^{-t} \to 0$	M1	Substitute into DE
	Giving $0 = 2 - 2y + 0$, so $y \to 1$	A1 **7**	
(ii)	Curve through origin with positive gradient	B1	
	With maximum at (ln4, 9/8)	B1	Follow their ln4
	With $y \to 1$ as $x \to \infty$	B1 **3**	Follow their (*C*)
4(i)	$\ddot{x} = 7\dot{x} + 6\dot{y} - 6e^{-3t}$	M1	Differentiate
	$= 7\dot{x} + 6(-12x - 10y + 5\sin t) - 6e^{-3t}$	M1	Substitute for $\dot{y}$
	$y = \frac{1}{6}(\dot{x} - 7x - 2e^{-3t})$	M1	y in terms of $x, \dot{x}, t$

	$\ddot{x}=7\dot{x}-72x-10(\dot{x}-7x-2e^{-3t})+30\sin t-6e^{-3t}$	M1		Substitute for *y*
	$\ddot{x}+3\dot{x}+2x=14e^{-3t}+30\sin t$	E1		Complete argument
			5	
(ii)	$x=ae^{-3t}-9\cos t+3\sin t$			
	$\dot{x}=-3ae^{-3t}+9\sin t+3\cos t$	M1		Differentiate twice
	$\ddot{x}=9ae^{-3t}+9\cos t-3\sin t$			
	In DE gives	M1		Substitute
	$9ae^{-3t}+9\cos t-3\sin t$			
	$+3(-3ae^{-3t}+9\sin t+3\cos t)$			
	$+2(ae^{-3t}-9\cos t+3\sin t)$			
	$=2ae^{-3t}+30\sin t$			
	So PI with $2a=14$	E1		Correct form shown
	$\Rightarrow a=7$	A1		
	AE $\alpha^2+3\alpha+2=0$	M1		Auxiliary equation
	$\alpha=-1,-2$	A1		
	CF $Ae^{-t}+Be^{-2t}$	F1		CF for their roots
	GS $x=Ae^{-t}+Be^{-2t}+7e^{-3t}-9\cos t+3\sin t$	F1		Their PI + their CF with two arbitrary constants
			8	
(iii)	$x=\frac{1}{6}(\dot{x}-7x-2e^{-3t})$	M1		y in terms of $x,\dot{x},t$
	$\dot{x}=-Ae^{-t}-2Be^{-2t}-21e^{-3t}+9\sin t+3\cos t$	M1		Differentiate GS for *x*
		F1		Follow their GS
	$y=-\frac{4}{3}Ae^{-t}-\frac{3}{2}Be^{-2t}-12e^{-3t}+11\cos t-2\sin t$	A1	**4**	cao
(iv)	$x\approx 3\sin t-9\cos t$	B1		Follow their *x*
	$y\approx 11\cos t-2\sin t$	B1		Follow their *y*
	$x=y\Rightarrow 11\cos t-2\sin t\approx 3\sin t-9\cos t$	M1		Equate
	$\Rightarrow 20\cos t\approx 5\sin t\Rightarrow \tan t\approx 4$	A1	**4**	Complete argument
(v)	Amplitude of $x\approx\sqrt{3^2+9^2}=3\sqrt{10}$	M1		Attempt both amplitudes
	Amplitude of $y\approx\sqrt{11^2+2^2}=5\sqrt{5}$	A1		One correct
	Ratio is $\frac{5}{6}\sqrt{2}$	A1	**3**	cao (accept reciprocal)

4758 Differential Equations (Written paper)

General Comments

The standard of work was generally very good, with many candidates demonstrating a clear understanding of the techniques required. Almost all candidates answered Questions 1 and 4, with Question 2 being the least popular choice. Candidates often produced accurate work in solving second order differential equations, but they seemed reluctant to adapt their standard method to take account of the information which was given to them in the question and which was intended to help them in Q.1(iv) and Q.4(ii).

With regard to graph sketching, it should be noted that the expectation in this unit is that any known information should be indicated on the sketch, i.e. given initial conditions and relevant results found earlier in the question. In addition, any particular features (e.g. oscillating, approaching an asymptote, bounds) should be clearly shown. Calculations beyond those already requested in the question are not required.

Comments on Individual Questions

1) Second order differential equation

(i) The method here was well-known. The main problem was in the choice of an appropriate trial function for the particular integral, with many opting for the incorrect y= Acos5t + Bsin5t. Candidates making this error were still able to gain method marks.

(ii) This was answered well by those candidates who had obtained a general solution in (i). Unfortunately those who persisted with the error noted in part (i) applied the initial conditions to what was in effect a complementary function.

(iii) Attempts at this sketch were reasonable, with candidates often gaining three out of the four marks. The amplitude was usually shown as constant, rather than increasing.

(iv) Most candidates gained full marks, but relatively few chose to take the hint, preferring to find the particular integral for themselves, rather than verify the given one.

(v) This was rarely answered correctly, with appropriate comment on the oscillatory nature and the boundedness of the two solutions being compared

2) First order differential equation

(i) Many candidates completed this correctly. Any loss of marks was due to either sign errors in the integration by parts or omission of an arbitrary constant.

(ii) Candidates were able to use the given approximations but then seemed to struggle with giving a clear explanation as to why the arbitrary constant was zero.

(iii) This was answered well by those who had been successful in part (ii).

(iv) Candidates were able to find the correct general solution to this differential equation, but as in part (ii) struggled to deal with the condition that y remains finite as x tends to zero

3) First order differential equations

(a)(i) This was well-answered by almost all who attempted it. The few who chose to ignore the request to find complementary function and particular integral and instead attempted an integrating factor method were always unsuccessful.

(ii) There were no problems with this.

(iii) A surprising number of candidates were unable to find the amplitude and fewer still produced a sketch which showed an oscillation of constant (and labelled) amplitude.

(b)(i) This question was different to most previously asked and it was very pleasing that the vast majority of candidates who attempted it were able to produce very good solutions.

(ii) This was well-answered, with candidates able to use their results to part (i) to produce good sketches.

4) Simultaneous differential equations

(i) The method here was applied with pleasing algebraic and numerical accuracy.

(ii) The majority of candidates proceeded to find the particular integral from scratch, rather than heeding the (helpful) advice in the question. No marks were lost for this, but time was wasted.

(iii) The method was known, but accuracy errors were abundant.

(iv) Candidates seemed not to realise what was required here. Some addressed the condition for large values of t, but did not see that they had then merely to equate their expressions for x and y.

(v) As in Question 3, there were problems in calculating amplitudes.