OCR
RECOGNISING ACHIEVEMENT

OXFORD CAMBRIDGE AND RSA EXAMINATIONS

Advanced Subsidiary General Certificate of Education
Advanced General Certificate of Education

MEI STRUCTURED MATHEMATICS 4755

Further Concepts for Advanced Mathematics (FP1)

Thursday **8 JUNE 2006** Morning 1 hour 30 minutes

Additional materials:
8 page answer booklet
Graph paper
MEI Examination Formulae and Tables (MF2)

TIME 1 hour 30 minutes

INSTRUCTIONS TO CANDIDATES

- Write your name, centre number and candidate number in the spaces provided on the answer booklet.
- Answer **all** the questions.
- You are permitted to use a graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets [ ] at the end of each question or part question.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.
- The total number of marks for this paper is 72.

This question paper consists of 3 printed pages and 1 blank page.

Section A (36 marks)

1 **(i)** State the transformation represented by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. [1]

(ii) Write down the 2×2 matrix for rotation through 90° anticlockwise about the origin. [1]

(iii) Find the 2×2 matrix for rotation through 90° anticlockwise about the origin, followed by reflection in the x-axis. [2]

2 Find the values of A, B, C and D in the identity

$$2x^3 - 3x^2 + x - 2 \equiv (x+2)(Ax^2 + Bx + C) + D.$$ [5]

3 The cubic equation $z^3 + 4z^2 - 3z + 1 = 0$ has roots α, β and γ.

(i) Write down the values of $\alpha + \beta + \gamma$, $\alpha\beta + \beta\gamma + \gamma\alpha$ and $\alpha\beta\gamma$. [3]

(ii) Show that $\alpha^2 + \beta^2 + \gamma^2 = 22$. [3]

4 Indicate, on separate Argand diagrams,

(i) the set of points z for which $|z-(3-\mathrm{j})| \leqslant 3$, [3]

(ii) the set of points z for which $1 < |z-(3-\mathrm{j})| \leqslant 3$, [2]

(iii) the set of points z for which $\arg(z-(3-\mathrm{j})) = \frac{1}{4}\pi$. [3]

5 **(i)** The matrix $\mathbf{S} = \begin{pmatrix} -1 & 2 \\ -3 & 4 \end{pmatrix}$ represents a transformation.

(*A*) Show that the point $(1, 1)$ is invariant under this transformation. [1]

(*B*) Calculate $\mathbf{S}^{-1}$. [2]

(*C*) Verify that $(1, 1)$ is also invariant under the transformation represented by $\mathbf{S}^{-1}$. [1]

(ii) Part **(i)** may be generalised as follows.

If (x, y) is an invariant point under a transformation represented by the non-singular matrix $\mathbf{T}$, it is also invariant under the transformation represented by $\mathbf{T}^{-1}$.

Starting with $\mathbf{T}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$, or otherwise, prove this result. [2]

6 Prove by induction that $3 + 6 + 12 + \ldots + 3 \times 2^{n-1} = 3(2^n - 1)$ for all positive integers n. [7]

Section B (36 marks)

7 A curve has equation $y = \dfrac{x^2}{(x-2)(x+1)}$.

(i) Write down the equations of the three asymptotes. [3]

(ii) Determine whether the curve approaches the horizontal asymptote from above or from below for

(A) large positive values of x,

(B) large negative values of x. [3]

(iii) Sketch the curve. [4]

(iv) Solve the inequality $\dfrac{x^2}{(x-2)(x+1)} > 0$. [3]

8 **(i)** Verify that $2 + \mathrm{j}$ is a root of the equation $2x^3 - 11x^2 + 22x - 15 = 0$. [5]

(ii) Write down the other complex root. [1]

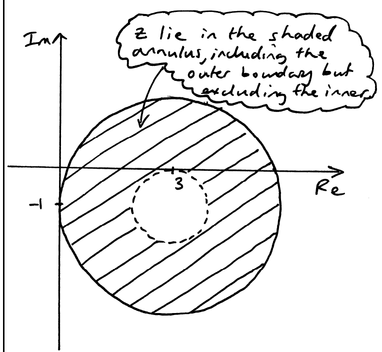
(iii) Find the third root of the equation. [4]

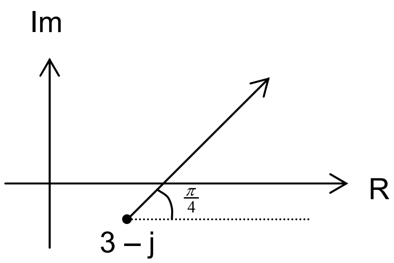
9 **(i)** Show that $r(r+1)(r+2) - (r-1)r(r+1) \equiv 3r(r+1)$. [2]

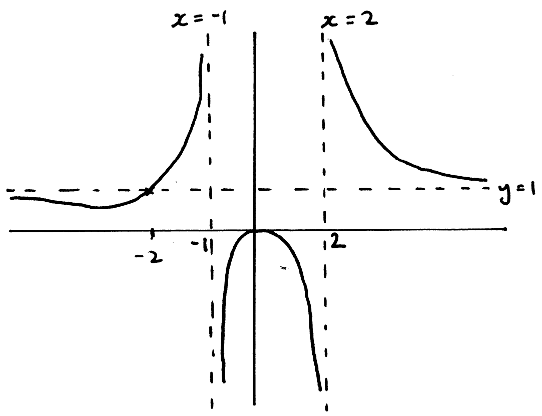
(ii) Hence use the method of differences to find an expression for $\sum_{r=1}^{n} r(r+1)$. [6]

(iii) Show that you can obtain the same expression for $\sum_{r=1}^{n} r(r+1)$ using the standard formulae for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^2$. [5]

Mark Scheme 4755
June 2006

Qu	Answer	Mark	Comment
Section A			
1 (i)	Reflection in the *x*-axis.	B1 **[1]**	
1(ii)	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	B1 **[1]**	
1(iii)	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$	M1 A1 c.a.o. **[2]**	Multiplication of their matrices in the correct order or B2 for correct matrix without working
2	$(x+2)(Ax^2+Bx+C)+D$ $= Ax^3+Bx^2+Cx+2Ax^2+2Bx+2C+D$ $= Ax^3+(2A+B)x^2+(2B+C)x+2C+D$ $\Rightarrow A=2,\ B=-7,\ C=15,\ D=-32$	M1 B1 B1 F1 F1 **OR** B5 **[5]**	Valid method to find all coefficients For *A* = 2 For D = -32 F1 for each of *B* and *C* For all correct
3(i)	$\alpha+\beta+\gamma=-4$ $\alpha\beta+\beta\gamma+\alpha\gamma=-3$ $\alpha\beta\gamma=-1$	B1 B1 B1 **[3]**	
3(ii)	$\alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\alpha\gamma)$ $=16+6=22$	M1 A1 E1 **[3]**	Attempt to use $(\alpha+\beta+\gamma)^2$ Correct Result shown
4 (i)	Argand diagram with solid circle, centre 3 – j, radius 3, with values of *z* on and within the circle clearly indicated as satisfying the inequality.	B1 B1 B1 **[3]**	Circle, radius 3, shown on diagram Circle centred on 3 - j Solution set indicated (solid circle with region inside)
4(ii)		B1 B1 **[2]**	 Hole, radius 1, shown on diagram Boundaries dealt with correctly

Qu	Answer	Mark	Comment
Section A (continued)			
4(iii)		B1 B1 B1 **[3]**	Line through their 3 – j Half line $\frac{\pi}{4}$ to real axis
5(i)	$\begin{pmatrix} -1 & 2 \\ -3 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\mathbf{S}^{-1} = \frac{1}{2}\begin{pmatrix} 4 & -2 \\ 3 & -1 \end{pmatrix}$ $\frac{1}{2}\begin{pmatrix} 4 & -2 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$	B1 M1, A1 E1 **[4]**	Attempt to divide by determinant and manipulate contents Correct
5(ii)	$\mathbf{T}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$ $\Rightarrow \mathbf{T}^{-1}\mathbf{T}\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{T}^{-1}\begin{pmatrix} x \\ y \end{pmatrix}$ $\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{T}^{-1}\begin{pmatrix} x \\ y \end{pmatrix}$	M1 A1 **[2]**	Pre-multiply by $\mathbf{T}^{-1}$ Invariance shown
6	$3+6+12+........+3\times 2^{n-1} = 3\left(2^n - 1\right)$ n = 1, LHS = 3, RHS = 3 Assume true for $n = k$ Next term is $3\times 2^{k+1-1} = 3\times 2^k$ Add to both sides $\text{RHS} = 3\left(2^k - 1\right) + 3\times 2^k$ $= 3\left(2^k - 1 + 2^k\right)$ $= 3\left(2\times 2^k - 1\right)$ $= 3\left(2^{k+1} - 1\right)$ But this is the given result with k + 1 replacing k. Therefore if it is true for k it is true for k + 1. Since it is true for k = 1, it is true for all positive integers n.	B1 E1 B1 M1 A1 E1 E1 **[7]**	Assuming true for k $(k + 1)^{\text{th}}$ term. Add to both sides Working must be valid Dependent on previous A1and E1 Dependent on B1 and previous E1
			Section A Total: 36

Section B			
7(i)	$x = 2$, $x = -1$ and $y = 1$	B1 B1B1 **[3]**	One mark for each
7(ii) (*A*) (*B*)	Large positive *x*, $y \to 1^+$ (from above) (e.g. consider $x = 100$) Large negative *x*, $y \to 1^-$ (from below) (e.g. consider $x = -100$)	M1 B1 B1 **[3]**	Evidence of method needed for M1
7(iii)	Curve 3 branches Correct approaches to horizontal asymptote Asymptotes marked Through origin 	B1 B1 B1 B1 **[4]**	With correct approaches to vertical asymptotes Consistent with their (i) and (ii) Equations or values at axes clear
7(iv)	$x < -1$, $x > 2$	B1B1, B1, **[3]**	s.c. 1 for inclusive inequalities Final B1 for all correct with no other solutions

8(i)	$(2+j)^2 = 3+4j$	B1	
	$(2+j)^3 = 2+11j$	B1	
	Substituting into $2x^3 - 11x^2 + 22x - 15$:	M1	Attempt at substitution
	$2(2+11j) - 11(3+4j) + 22(2+j) - 15$	A1	Correctly substituted
	$= 4 + 22j - 33 - 44j + 44 + 22j - 15$ $= 0$ So 2 + j is a root.	A1 **[5]**	Correctly cancelled (Or other valid methods)
8(ii)	2 - j	B1 **[1]**	
8(iii)	$(x-(2+j))(x-(2-j))$ $=(x-2-j)(x-2+j)$	M1	Use of factor theorem
	$= x^2 - 2x + jx - 2x + 4 - 2j - jx + 2j + 1$ $= x^2 - 4x + 5$	A1	
	$(x^2 - 4x + 5)(ax+b) = 2x^3 - 11x^2 + 22x - 15$ $(x^2 - 4x + 5)(2x-3) = 2x^3 - 11x^2 + 22x - 15$	M1	Comparing coefficients or long division
	$(2x-3) = 0 \Rightarrow x = \frac{3}{2}$	A1 **[4]**	Correct third root
	OR		
	Sum of roots = $\frac{11}{2}$ **or** product of roots = $\frac{15}{2}$	M1 A1	
	leading to $\alpha + 2 + j + 2 - j = \frac{11}{2}$	M1	
	$\Rightarrow \alpha = \frac{3}{2}$	A1 **[4]**	
	or $\alpha(2+j)(2-j) = \frac{15}{2}$	M1 A1	
	$\Rightarrow 5\alpha = \frac{15}{2} \Rightarrow \alpha = \frac{3}{2}$	M1 A1 **[4]**	(Or other valid methods)

9(i)	$r(r+1)(r+2)-(r-1)r(r+1)$ $\equiv (r^2+r)(r+2)-r^3-r$ $\equiv r^3+2r^2+r^2+2r-r^3+r$ $\equiv 3r^2+3r \equiv 3r(r+1)$	M1 E1 **[2]**	Accept '=' in place of '≡' throughout working Clearly shown
9(ii)	$\sum_{r=1}^{n} r(r+1)$ $=\frac{1}{3}\sum_{r=1}^{n}[r(r+1)(r+2)-(r-1)r(r+1)]$ $=\frac{1}{3}[(1\times2\times3-0\times1\times2)+(2\times3\times4-1\times2\times3)+$ $(3\times4\times5-2\times3\times4)+......$ $+(n(n+1)(n+2)-(n-1)n(n+1))]$ $=\frac{1}{3}n(n+1)(n+2)$ or equivalent	M1 M1 A2 M1 A1 **[6]**	Using identity from (i) Writing out terms in full At least 3 terms correct (minus 1 each error to minimum of 0) Attempt at eliminating terms (telescoping) Correct result
9(iii)	$\sum_{r=1}^{n} r(r+1)=\sum_{r=1}^{n} r^2+\sum_{r=1}^{n} r$ $=\frac{1}{6}n(n+1)(2n+1)+\frac{1}{2}n(n+1)$ $=\frac{1}{6}n(n+1)[(2n+1)+3]$ $=\frac{1}{6}n(n+1)(2n+4)$ $=\frac{1}{3}n(n+1)(n+2)$ or equivalent	B1 B1 M1 A1 E1 **[5]**	Use of standard sums (1 mark each) Attempt to combine Correctly simplified to match result from (ii)
			Section B Total: 36
			Total: 72

4755 - Further Concepts for Advanced Mathematics (FP1)

General Comments

Many strong candidates took this paper and many scored very highly. However, an encouraging number of centres had candidates scoring marks across the full grade range. This seems to indicate that centres are beginning to enter candidates across a wider ability range, taking advantage of the new style AS Further Mathematics and resulting in an increasing entry.

A small proportion of candidates were not well prepared for a paper of this nature.

The overall standard of scripts was very good and many candidates showed a pleasing level of algebraic competence. However, missing brackets, imprecise explanations and poor use of notation were apparent on a significant number of scripts.

Candidates must be sure to label their diagrams clearly. Ambiguous labelling was particularly apparent in question 4.

Some candidates, both weak and strong, seemed to spend an inappropriate amount of time for questions worth few marks. This was particularly the case for 5 (ii).

Comments on Individual Questions

1) **Matrices**
Surprisingly this was one of the least well answered questions on the paper. Most candidates got part (i) correct but many gave an incorrect matrix for part (ii). A common mistake in part (iii) was to multiply the two matrices in the wrong order.

2) **Roots of a cubic equation**
Both parts of this question were well answered. A few candidates made sign errors in part (i). Most of those who attempted part (ii) were successful but some did not attempt it.

3) **Identity**
This question was answered correctly by almost all candidates.

4) **Loci of complex numbers on the Argand diagram**
While most candidates scored fairly well on this question, a significant few seemed totally unprepared for a question of this type.

In part (i) the commonest mistakes were incorrect or missing shading and the use of an incorrect point for the centre.

The commonest mistake in part (ii) was to fail to exclude the circumference of the inner circle.

Part (iii) produced many mistakes: drawing a whole line rather than just half; drawing it at the wrong angle and using the wrong end point. Several candidates also drew circles or sectors.

Many candidates did not label their diagrams clearly, particularly the boundaries in part (ii).

5) **Invariant points**

Most candidates got part (i) right but there were also several mistakes on the inverse matrix such as multiplying by the determinant and misplacing the numbers inside the matrix.

Part (ii) proved to be a good discriminator for the stronger candidates, many of whom got it right whilst most candidates either omitted it, or spent considerable time without making any worthwhile progress. A common mistake among some stronger candidates was to post-multiply by $\mathbf{T}^{-1}$ instead of pre-multiplying, or to pre-multiply one side of the equation and post-multiply the other. Several candidates tried to work from the general matrix $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$; although this can lead to a correct answer, the working is not simple and candidates who tried this method were invariably unsuccessful. There is a lesson to be learnt from the contrast between the simplicity of using matrix algebra and the difficulty of any other method.

6) **Proof by induction**

The general standard of the answers to this question was better than in earlier papers. However, some candidates did not convey the logic of the method, for example writing "Let $n = k$" instead of "Assume the result is true for $n = k$". In the central working 3.2^k and 3.2^{k+1} were often changed into 6^k and 6^{k+1} and it was apparent that several candidates were not really sure why $3.2^k - 3 + 3.2^k$ was equal to $3(2^{k+1} - 1)$. Many failed to explain the steps in their proof adequately and so did not convince that they understood the structure of the proof. Since this question is about proof, such errors incurred a substantial mark penalty.

7) **Graph**

By far the majority of candidates were well prepared for this question. It was unusual, for example, to see the horizontal asymptote stated as $y = 0$ instead of $y = 1$ in part (i).

(i) The vast majority of candidates answered this correctly.

(ii) A common mistake in part (ii) was to write $\frac{\infty^2}{\infty^2}$. Many candidates stated that for large negative x the curve approached the asymptote from above, even though their working had shown them otherwise. Many candidates showed no workings to justify their answers and so forfeited at least one mark.

(iii) Most candidates obtained the correct branches for the curve but a significant few drew the left hand branch appearing from the wrong end of the $x = -1$ vertical asymptote. This may have been because candidates mistakenly believed that a graph cannot cross an asymptote, whereas this graph crosses the horizontal asymptote from above, before tending to it from below for large negative values of x.

A few candidates lost marks by not labelling the asymptotes on their graphs.

(iv) In part (iv), many candidates made the mistake of giving the answer as $-1 > x > 2$. It cannot, of course be expressed as a single compound inequality of this type.

Many candidates spent a considerable time on workings for this part of the question when they could have written the answers down from looking at their graph.

8) **Complex roots of an equation**
This question was very well answered with many candidates obtaining full marks.

(i) In part (i), most candidates knew to substitute the given root 2 + j into the equation and to expect an answer of zero. Those who made careless mistakes tended to go back and correct them.

(ii) Very few candidates failed to get part (ii) right.

(iii) Part (iii) was also well answered. About half the candidates answered using the sum of the roots or the product of the roots, and about half found the quadratic factor and then the linear one. Of those using the factor method, some candidates gave $(x-3)$ rather than $(2x-3)$ as the third factor. It was also not uncommon for candidates to say that $((x-2-\mathrm{j})\times(x-2+\mathrm{j}) = x^2-4x+3$) instead of x^2-4x+5 leading to a third factor of $(2x-5)$. Several candidates stated that their third factor was the root.

9) **Summation of series**
This question was well answered. Many of the stronger candidates obtained full marks on it. However, it was clear that quite a few candidates were in a rush to finish.

(i) A substantial number of candidates made sign errors and then claimed to have arrived at the given answer. A number showed scant regard for the need for brackets and paid the penalty by claiming $3r^2 + r$ was equal to $3r(r+1)$.

(ii) The method of differences was generally well done, though there was much slack use of brackets. A significant number of candidates missed out the factor of $\frac{1}{3}$. Some candidates gave their answer in terms of r instead of n. Several candidates did not obey the instruction to use the method of differences and so were awarded no marks for this part.

(iii) Most candidates knew just what to do and came out with the required result. There were a few algebraic errors, especially involving fractions and factorisation. A number of candidates spent considerable time trying to match an incorrect result from part (ii), in some cases having actually achieved the right answer in part (iii). A small number of candidates had difficulty recalling the correct expression for $\sum r$.