ADVANCED GCE UNIT **4763/01**

MATHEMATICS (MEI)

Mechanics 3

WEDNESDAY 10 JANUARY 2007

Afternoon
Time: 1 hour 30 minutes

Additional materials:
Answer booklet (8 pages)
Graph paper
MEI Examination Formulae and Tables (MF2)

INSTRUCTIONS TO CANDIDATES

- Write your name, centre number and candidate number in the spaces provided on the answer booklet.
- Answer **all** the questions.
- You are permitted to use a graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.
- The acceleration due to gravity is denoted by g m s^{-2}. Unless otherwise instructed, when a numerical value is needed, use $g = 9.8$.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets [ ] at the end of each question or part question.
- The total number of marks for this paper is 72.

ADVICE TO CANDIDATES

- Read each question carefully and make sure you know what you have to do before starting your answer.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.

This document consists of **6** printed pages and **2** blank pages.

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1 **(i)** Write down the dimensions of velocity, acceleration and force. [3]

The force F of gravitational attraction between two objects with masses m_1 and m_2, at a distance r apart, is given by

$$F = \frac{Gm_1m_2}{r^2}$$

where G is the universal constant of gravitation.

(ii) Show that the dimensions of G are $\mathrm{M}^{-1}\mathrm{L}^3\mathrm{T}^{-2}$. [2]

(iii) In SI units (based on the kilogram, metre and second) the value of G is 6.67×10^{-11}.

Find the value of G in imperial units based on the pound (0.4536 kg), foot (0.3048 m) and second. [3]

(iv) For a planet of mass m and radius r, the escape velocity v from the planet's surface is given by

$$v = \sqrt{\frac{2Gm}{r}}.$$

Show that this formula is dimensionally consistent. [3]

(v) For a planet in circular orbit of radius R round a star of mass M, the time t taken to complete one orbit is given by

$$t = kG^{\alpha}M^{\beta}R^{\gamma}$$

where k is a dimensionless constant.

Use dimensional analysis to find α, β and γ. [5]

2 **(a)** A light inextensible string has length 1.8 m. One end of the string is attached to a fixed point O, and the other end is attached to a particle of mass 5 kg. The particle moves in a complete vertical circle with centre O, so that the string remains taut throughout the motion. Air resistance may be neglected.

(i) Show that, at the highest point of the circle, the speed of the particle is at least $4.2\,\mathrm{m\,s^{-1}}$. [3]

(ii) Find the least possible tension in the string when the particle is at the lowest point of the circle. [5]

(b) Fig. 2 shows a hollow cone mounted with its axis of symmetry vertical and its vertex V pointing downwards. The cone rotates about its axis with a constant angular speed of $\omega\,\mathrm{rad\,s^{-1}}$. A particle P of mass 0.02 kg is in contact with the rough inside surface of the cone, and does not slip. The particle P moves in a horizontal circle of radius 0.32 m. The angle between VP and the vertical is θ.

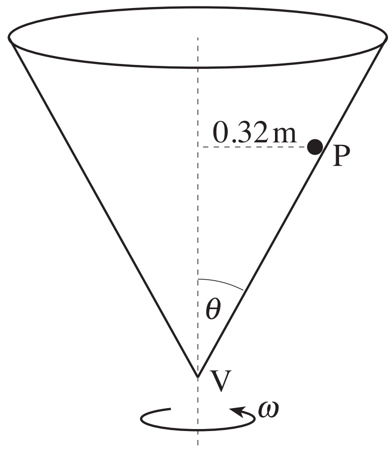

Fig. 2

In the case when $\omega = 8.75$, there is no frictional force acting on P.

(i) Show that $\tan\theta = 0.4$. [4]

Now consider the case when ω takes a constant value greater than 8.75.

(ii) Draw a diagram showing the forces acting on P. [2]

(iii) You are given that the coefficient of friction between P and the surface is 0.11. Find the maximum possible value of ω for which the particle does not slip. [6]

3 Ben has mass 60 kg and he is considering doing a bungee jump using an elastic rope with natural length 32 m. One end of the rope is attached to a fixed point O, and the other end is attached to Ben. When Ben is supported in equilibrium by the rope, the length of the rope is 32.8 m.

To predict what will happen, Ben is modelled as a particle B, the rope is assumed to be light, and air resistance is neglected. B is released from rest at O and falls vertically. When the rope becomes stretched, x m denotes the extension of the rope.

(i) Find the stiffness of the rope. [2]

(ii) Use an energy argument to show that, when B comes to rest instantaneously with the rope stretched,

$$x^2 - 1.6x - 51.2 = 0.$$

Hence find the length of the rope when B is at its lowest point. [6]

(iii) Show that, while the rope is stretched,

$$\frac{d^2x}{dt^2} + 12.25x = 9.8$$

where t is the time measured in seconds. [4]

(iv) Find the time taken for B to travel between the equilibrium position ($x = 0.8$) and the lowest point. [3]

(v) Find the acceleration of B when it is at the lowest point, and comment on the implications for Ben. [3]

4 In this question, a is a constant with $a > 1$.

Fig. 4 shows the region bounded by the curve $y = \dfrac{1}{x^2}$ for $1 \leqslant x \leqslant a$, the x-axis, and the lines $x = 1$ and $x = a$.

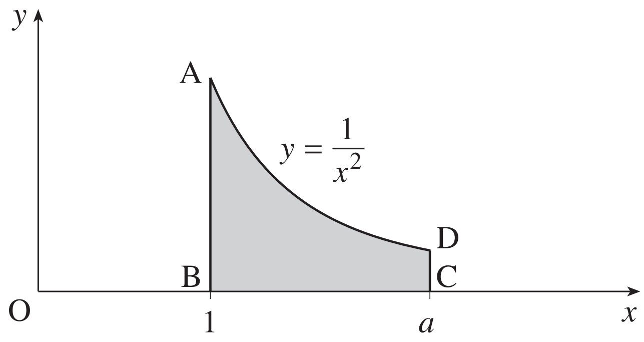

Fig. 4

This region is occupied by a uniform lamina ABCD, where A is $(1, 1)$, B is $(1, 0)$, C is $(a, 0)$ and D is $\left(a, \dfrac{1}{a^2}\right)$. The centre of mass of this lamina is $(\overline{x}, \overline{y})$.

(i) Find $\overline{x}$ in terms of a, and show that $\overline{y} = \dfrac{a^3 - 1}{6(a^3 - a^2)}$. [8]

(ii) In the case $a = 2$, the lamina is freely suspended from the point A, and hangs in equilibrium. Find the angle which AB makes with the vertical. [3]

The region shown in Fig. 4 is now rotated through 2π radians about the x-axis to form a uniform solid of revolution.

(iii) Find the x-coordinate of the centre of mass of this solid of revolution, in terms of a, and show that it is less than 1.5. [7]

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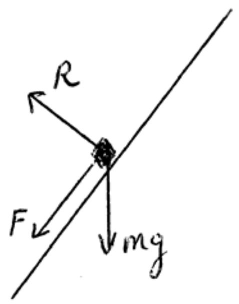
Mark Scheme 4763
January 2007

1 (i)	[Velocity] $=\mathrm{L\,T^{-1}}$ [Acceleration] $=\mathrm{L\,T^{-2}}$ [Force] $=\mathrm{M\,L\,T^{-2}}$	B1 B1 B1 **3**	*Deduct 1 mark if answers given as* $\mathrm{m\,s^{-1}},\ \mathrm{m\,s^{-2}},\ \mathrm{kg\,m\,s^{-2}}$
(ii)	$[G]=\dfrac{[F][r^2]}{[m_1][m_2]}=\dfrac{(\mathrm{M\,L\,T^{-2}})(\mathrm{L^2})}{\mathrm{M^2}}$ $=\mathrm{M^{-1}\,L^3\,T^{-2}}$	M1 E1 **2**	
(iii)	$G=6.67\times10^{-11}\times0.4536\times\dfrac{1}{(0.3048)^3}$ $=1.07\times10^{-9}\quad(\ \mathrm{lb^{-1}\,ft^3\,s^{-2}}\ )$	M1M1 A1 **3**	For $\times0.4536$ and $\times\dfrac{1}{(0.3048)^3}$ *SC* Give M1 for $6.67\times10^{-11}\times\dfrac{1}{0.4536}\times(0.3048)^3$ $(\ =4.16\times10^{-12}\ )$
(iv)	$[\mathrm{RHS}]=\sqrt{\dfrac{(\mathrm{M^{-1}\,L^3\,T^{-2}})(\mathrm{M})}{\mathrm{L}}}$ $=\sqrt{\mathrm{L^2\,T^{-2}}}=\mathrm{L\,T^{-1}}$ which is the same as [LHS]	M1A1 E1 **3**	
(v)	$\mathrm{T}=(\mathrm{M^{-1}\,L^3\,T^{-2}})^{\alpha}\,\mathrm{M}^{\beta}\,\mathrm{L}^{\gamma}$ Powers of M: $-\alpha+\beta=0$ of L: $3\alpha+\gamma=0$ of T: $-2\alpha=1$ $\alpha=-\frac{1}{2},\quad\beta=-\frac{1}{2},\quad\gamma=\frac{3}{2}$	M1 M1 A1 M1 A1 **5**	At least two equations Three correct equations Obtaining at least one of $\alpha,\ \beta,\ \gamma$

2(a)(i)	At the highest point, $T + 5 \times 9.8 = 5 \times \dfrac{v^2}{1.8}$ For least speed, $T = 0$, $v^2 = 1.8 \times 9.8$ Speed is at least $4.2\ \text{m s}^{-1}$	M1 A1 E1 **3**	Using acceleration $v^2/1.8$ *T* may be omitted
(ii)	For least tension, speed at top is $4.2\ \text{m s}^{-1}$ By conservation of energy, $\frac{1}{2} \times 5 \times (w^2 - 4.2^2) = 5 \times 9.8 \times 3.6$ $w^2 = 88.2 \quad (w = 9.39)$ $T - 5 \times 9.8 = 5 \times \dfrac{88.2}{1.8}$ Tension is at least 294 N	M1 A1 M1 A1 ft A1 **5**	Energy equation with 3 terms Equation of motion with 3 terms
(b)(i)	$R\sin\theta = 0.02 \times 9.8$ $R\cos\theta = 0.02 \times 0.32 \times 8.75^2$ $\tan\theta = \dfrac{0.02 \times 9.8}{0.02 \times 0.32 \times 8.75^2} = 0.4$	B1 M1 A1 E1 **4**	Using acceleration 0.32×8.75^2 *SC* If $\sin\theta$ and $\cos\theta$ interchanged, award B0M1A1E0
(ii)		B1 B1 **2**	For *R* and *mg* For *F* acting down the slope
(iii)	$R\sin\theta = 0.02 \times 9.8 + F\cos\theta$ $R\cos\theta + F\sin\theta = 0.02 \times 0.32\,\omega^2$ For maximum ω, $F = \mu R$ $R(\sin\theta - \mu\cos\theta) = 0.02 \times 9.8$ $R(\cos\theta + \mu\sin\theta) = 0.02 \times 0.32\,\omega^2$ $\omega^2 = \dfrac{9.8(\cos\theta + \mu\sin\theta)}{0.32(\sin\theta - \mu\cos\theta)} = \dfrac{9.8(1 + \mu\tan\theta)}{0.32(\tan\theta - \mu)}$ $= \dfrac{9.8(1 + 0.11 \times 0.4)}{0.32(0.4 - 0.11)}$ $\omega = 10.5$	M1 A1 A1 M1 M1 A1 cao **6**	Resolving *F* and *R* [or *mg* and accn] *Can give A1A1 for sin / cos interchanged consistent with (i)* *Dependent on first M1* Obtaining a numerical value for ω^2 *Dependent on M1M1*

3 (i)	$k \times 0.8 = 60 \times 9.8$ Stiffness is 735 N m^{-1}	M1 A1 **2**	
(ii)	Loss of PE is $60 \times 9.8(32 + x)$ Gain in EE is $\frac{1}{2} \times 735x^2$ $\frac{1}{2} \times 735x^2 = 60 \times 9.8(32 + x)$ $x^2 = 1.6(32 + x)$ $x^2 - 1.6x - 51.2 = 0$ $(x - 8)(x + 6.4) = 0$ $x = 8$ Length of rope is 40 m	B1 B1 M1 E1 M1 A1 **6**	*If x is measured from equilibrium position, treat as MR* Obtaining a value of *x*
(iii)	Tension $T = 735x$ $mg - T = m\frac{\mathrm{d}^2x}{\mathrm{d}t^2}$ $60 \times 9.8 - 735x = 60\frac{\mathrm{d}^2x}{\mathrm{d}t^2}$ $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 12.25x = 9.8$	B1 M1 A1 E1 **4**	 Equation of motion with 3 terms
(iv)	SHM with $\omega^2 = 12.25$ ($\omega = 3.5$) Time taken is $\frac{1}{4} \times \frac{2\pi}{\omega}$ $= \frac{1}{7}\pi = 0.449 \text{ s}$	M1 M1 A1 **3**	 or $\omega t = \frac{1}{2}\pi$
(v)	When $x = 8$, $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = 9.8 - 12.25 \times 8$ $= -88.2$ Acceleration is 88.2 m s^{-2} (upwards) This acceleration (9*g*) is too large for comfort	M1 A1 B1 **3**	or $735 \times 8 - 60 \times 9.8 = 60a$

4 (i)	Area is $\int_1^a \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^a$	M1		
	$= 1 - \frac{1}{a}$	A1		
	$\int x\,y\,dx = \int_1^a \frac{1}{x}\,dx \quad (= \ln a)$	M1		
	$\bar{x} = \frac{\int x\,y\,dx}{\int y\,dx}$	M1		
	$= \frac{\ln a}{1 - \frac{1}{a}} \quad \left(= \frac{a \ln a}{a-1} \right)$	A1		
	$\int \frac{1}{2} y^2\,dx = \int_1^a \frac{1}{2x^4}\,dx = \left[-\frac{1}{6x^3}\right]_1^a$ $= \frac{1}{6}\left(1 - \frac{1}{a^3}\right)$	M1		Condone omission of $\frac{1}{2}$
	$\bar{y} = \frac{\int \frac{1}{2} y^2\,dx}{\int y\,dx}$	M1		($\frac{1}{2}$ needed for this mark)
	$= \frac{\frac{1}{6}\left(1 - \frac{1}{a^3}\right)}{1 - \frac{1}{a}} = \frac{a^3 - 1}{6(a^3 - a^2)}$	E1	**8**	
(ii)	When $a = 2$, $\bar{x} = 2\ln 2$, $\bar{y} = \frac{7}{24}$			
	$\tan\theta = \frac{\bar{x} - 1}{1 - \bar{y}}$	M1		CM vertically below A
	$= \frac{2\ln 2 - 1}{1 - \frac{7}{24}}$	A1		Correct expression for $\tan\theta$ or $\tan(90 - \theta)$
	$\theta = 28.6°$	A1	**3**	

(iii)	Volume is $\int \pi y^2\,dx \quad = \pi \int_1^a \frac{1}{x^4}\,dx$	M1	π may be omitted throughout
	$= \pi \left[-\frac{1}{3x^3} \right]_1^a = \frac{\pi}{3}\left(1 - \frac{1}{a^3}\right)$	A1	
	$\int \pi x y^2\,dx \quad = \pi \int_1^a \frac{1}{x^3}\,dx = \pi \left[-\frac{1}{2x^2} \right]_1^a$	M1	
	$= \frac{\pi}{2}\left(1 - \frac{1}{a^2}\right)$		
	$\bar{x} = \frac{\int \pi x y^2\,dx}{\int \pi y^2\,dx}$	M1	
	$= \frac{\frac{\pi}{2}\left(1 - \frac{1}{a^2}\right)}{\frac{\pi}{3}\left(1 - \frac{1}{a^3}\right)} = \frac{3(a^3 - a)}{2(a^3 - 1)}$	A1	Any correct form
	Since $a > 1, \quad a^3 - a < a^3 - 1$		
	Hence $\bar{x} < \frac{3}{2}$, i.e. $\bar{x} < 1.5$	M1	or $\bar{x} \to 1.5$ as $a \to \infty$
		E1 **7**	Fully convincing argument

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General Comments

This paper was found to be somewhat more difficult than the 2006 papers, but the majority of candidates showed that they had a good understanding of most of the topics examined. About 30% of the candidates scored 60 marks or more (out of 72), and only about 15% scored less than half marks.

Comments on Individual Questions

1 **Dimensional Analysis**

This question was very well answered, with the majority of candidates scoring either 13 or 14 marks (out of 16). All parts were usually answered correctly, except for the change of units in part (iii); the conversion factor was very often the reciprocal of the correct one, and the given value (6.67×10^{-11}) was often omitted from the calculation.

2 **Circular Motion**

There was a much greater spread of marks on this question, and the average mark was about 13 (out of 20).
The vertical circle problem in part (a) was usually answered correctly, although the weight was often omitted from the calculation in part (ii), and a few assumed that the speed remained constant.

Part (b) was a horizontal circle problem, and most candidates were able to obtain the given result in part (i). The force diagram in part (ii) was usually correct, although the frictional force was often in the wrong direction, and sometimes omitted. The final part (iii) was found to be quite challenging. There were many excellent solutions; those who wrote down the vertical and horizontal equations of motion usually made substantial progress and often obtained the correct answer. However, a substantial number tried resolving in inappropriate directions and were not able to score any marks.

3 **Elasticity and Simple Harmonic Motion**

The average mark on this question was about 13 (out of 18).

Part (i) was usually answered correctly, although some candidates confused stiffness with the modulus of elasticity.

Part (ii) was generally well done (but some measured x from the equilibrium position and so could not obtain the required quadratic equation), and most candidates were able to obtain the given result in part (iii).

Part (iv) was found quite difficult. Those who recognised that the equation in part (iii) implied simple harmonic motion usually used $x = A\sin\omega t$ but the values of A and/or x were often incorrect; few seemed to realise that what was required was just one quarter of the period. A surprising number of candidates tried to apply constant acceleration formulae in this part.

In part (v), the acceleration was usually found correctly, and many candidates expressed concern for Ben's safety.

4 **Centres of Mass**

The principles involved in finding centres of mass by integration were very well understood, but the work was often spoilt by careless errors such as dropped minus signs, and powers of *a* going astray; the average mark was about 12 (out of 18). More serious errors, such as integrating $\frac{1}{x^4}$ to obtain $\frac{-1}{5x^5}$, or even $\ln(x^4)$, were also quite common.

In part (ii) the angle was often calculated as $\tan^{-1}\left(\frac{\bar{x}}{1-\bar{y}}\right)$ instead of $\tan^{-1}\left(\frac{\bar{x}-1}{1-\bar{y}}\right)$.
In part (iii), showing that $\bar{x} < 1.5$ proved to be quite challenging, even for those who had correctly obtained $\bar{x} = \frac{3(a^3 - a)}{2(a^3 - 1)}$. It was only necessary to observe that $a > 1$; but many candidates just substituted in a single value of *a* (usually $a = 2$), and some stated that $\bar{x} \to 1.5$ as $a \to \infty$ without saying that $\bar{x}$ is an increasing function of *a*.