OCR
RECOGNISING ACHIEVEMENT

OXFORD CAMBRIDGE AND RSA EXAMINATIONS

Advanced Subsidiary General Certificate of Education
Advanced General Certificate of Education

MEI STRUCTURED MATHEMATICS 4777

Numerical Computation

Wednesday **21 JUNE 2006** Afternoon 2 hours 30 minutes

Additional materials:
8 page answer booklet
Graph paper
MEI Examination Formulae and Tables (MF2)

TIME 2 hours 30 minutes

INSTRUCTIONS TO CANDIDATES

- Write your name, centre number and candidate number in the spaces provided on the answer booklet.
- Answer any **three** questions.
- Additional sheets, including computer print-outs, should be fastened securely to the answer booklet.

COMPUTER RESOURCES

- Candidates will require access to a computer with a spreadsheet program and suitable printing facilities during the examination..

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets [ ] at the end of each question or part question.
- In each of the questions you are required to write spreadsheet routines to carry out various numerical analysis processes. You should note the following points.
- You will not receive credit for using any numerical analysis functions which are provided within the spreadsheet. For example, many spreadsheets provide a solver routine; you will not receive credit for using this routine when asked to write your own procedure for solving an equation.

  You may use the following built-in mathematical functions: square root, sin, cos, tan, arcsin, arccos, arctan, ln, exp.
- For each question you attempt, you should submit print-outs showing the spreadsheet routine you have written and the output it generates. It will be necessary to print out the formulae in the cells as well as the values in the cells.

  You are not expected to print out and submit everything your routine produces, but you are required to submit sufficient evidence to convince the examiner that a correct procedure has been used.
- The total number of marks for this paper is 72.

This question paper consists of 5 printed pages and 3 blank pages.

1 **(i)** A sequence of numbers $x_1, x_2, x_3, \ldots$ is such that

$$x_{r+1} - \alpha \approx k(x_r - \alpha)$$

for some constants k and α.

Show that α may be estimated as $\dfrac{x_1^2 - x_0 x_2}{2x_1 - x_0 - x_2}$. [5]

(ii) An attempt is made to solve the equation $e^x = \tan x$ using the iterative formula

$$x_{r+1} = \ln(\tan x_r).$$

Show that the equation has a root in the interval $[1, 1.5]$. Demonstrate that the given iteration diverges for starting values in this interval.

Use the method based on the formula obtained in part **(i)** to obtain the root correct to 5 decimal places. [11]

(iii) Show that the equation $e^{-x} = \tan x$ has a root that is just slightly greater than π. Demonstrate that the iteration

$$x_{r+1} = -\ln(\tan x_r)$$

fails to converge to this root.

Show that the approach used in part **(ii)** will give convergence to the required root, but that a very accurate starting value is required. Give the root correct to 5 decimal places. [8]

2 **(i)** Explain briefly the advantage, relative to other methods of interpolation, of using divided differences. [2]

The function $f(x)$ has known values as given, correct to 2 decimal places, in the table.

x	1	1.5	2	2.5	3	3.5	4	4.5	5
$f(x)$	–3.00		−6.50	−8.03		−6.66	−2.25	5.65	

(ii) Draw up a divided difference table to produce a sequence of estimates, linear, quadratic, cubic and quartic, for $f(1.5)$. Discuss briefly the accuracy to which it is possible to estimate $f(1.5)$. [10]

(iii) Modify your routine from part **(i)** to produce estimates of

(*A*) f(3),

(*B*) f(5).

In each case discuss briefly the likely accuracy of your estimate. [6]

(iv) There is a root of the equation $f(x) = 0$ at x just greater than 4. Modify your routine to estimate values of $f(x)$ near $x = 4$. Hence, by trial and error, determine the root correct to 2 decimal places. [6]

[Turn over

3 The second order differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 2y^2 = 2e^{-kx}$$

with initial conditions $x = 0$, $y = 0$, $\frac{dy}{dx} = 0$, is to be solved, for various values of k, using finite difference methods. The value of y when $x = 1$ is required. This value is denoted by β.

(i) Consider first the case $k = 1$.

Show that, in the usual notation,

$$y_{r+1} = \frac{1}{1-2h}(2h^2e^{-x_r} - 2h^2y_r^2 + 2y_r - (1+2h)y_{r-1}),$$

and that

$$y_1 = h^2.$$

Show that, with $h = 0.1$, the estimate of β is a little greater than 4.25.

Obtain further estimates of β for $h = 0.05, 0.025, 0.0125$. Hence demonstrate that the method has second order convergence. Determine β correct to 2 decimal places. [17]

(ii) Modify the routines developed in part **(i)** to find estimates of β, correct to 1 decimal place, for $k = -5, -4, \ldots, 4, 5$. Use the spreadsheet to produce a graph of β as a function of k. [7]

4 **(i)** A set of simultaneous linear equations are to be solved using the Gauss-Seidel iterative method. Explain what diagonal dominance is, and how it relates to the convergence of the method.

Show by means of the equations with augmented matrix

$$\left(\begin{array}{ccc|c} 5 & 3 & 3 & 1 \\ 4 & 7 & 4 & 1 \\ 5 & 5 & 9 & 1 \end{array}\right)$$

that diagonal dominance is *not* a necessary condition. [7]

(ii) Modify the routine developed in part **(i)** to solve the equations with augmented matrix

$$\left(\begin{array}{ccc|c} 6-a & 3 & 3 & 1 \\ 4 & 8-a & 4 & 1 \\ 5 & 5 & 10-a & 1 \end{array}\right)$$

for user-specified values of a.

Demonstrate that the Gauss-Seidel iteration converges for $a = 3$ but diverges for $a = 4$. Determine to 1 decimal place the largest value of a for which the Gauss-Seidel iteration converges. [11]

(iii) Modify the routine in part **(ii)** so that it now implements the Gauss-Jacobi method. Show that the iteration now does *not* converge for $a = 0$. Explain how this result relates to the condition of diagonal dominance. [6]

Mark Scheme 4777
June 2006

MEI Numerical Computation (4777) June 2006 **Mark scheme**

1

(i) $(x_2 - \alpha) / (x_1 - \alpha) = (x_1 - \alpha) / (x_0 - \alpha)$ to eliminate k **[M1A1A1]**

convincing algebra to required result. **[A1A1]**

[subtotal 5]

(ii)

x	1	1.5		
exp(x) - tan(x)	1.160874	-9.61973	change of sign (and no asymptote)	**[M1A1]**

Examples of divergence:

r	0	1	2	3	4	5	6	
x_r	1	0.443023	-0.74554	#NUM!	#NUM!	#NUM!	#NUM!	
x_r	1.25	1.101797	0.67982 1	-0.21274	#NUM!	#NUM!	#NUM!	
x_r	1.5	2.646275	#NUM!	#NUM!	#NUM!	#NUM!	#NUM!	**[M1A1A1]**

x_r	1.25	1.101797	0.67982 1	α: 1.330227	**[M1A1]**
x_r	1.330227	1.405193	1.78895 9	α: 1.312029	**[M1A1]**
x_r	1.312029	1.329149	1.40054	α: 1.306628	
x_r	1.306628	1.307521	1.31106 9	α: 1.306328	
x_r	1.306328	1.30633	1.30634	α: 1.306327	
x_r	1.306327	1.306327	1.30632 7	α: 1.306328	**[M1]**
				1.30633 to 5 dp	**[A1]**

[subtotal 11]

(iii)

x	3.142	3.2		
exp(-x) - tan(x)	0.042789	-0.01771	change of sign	**[M1A1]**

Examples of divergence:

r	0	1	2	3	4	5	6	
x_r	3.142	7.805847	-3.03297	2.21594 3	#NUM!	#NUM!	#NUM!	
x_r	3.2	2.839176	#NUM!	#NUM!	#NUM!	#NUM!	#NUM!	**[M1A1]**

Eg:

x_r	3.18	3.259015	2.13737	α: 3.1852	
x_r	3.1852	3.131898	#NUM!	α: #NUM!	
x_r	#NUM!	#NUM!	#NUM!	α: #NUM!	
x_r	#NUM!	#NUM!	#NUM!	α: #NUM!	**[M1A1]**

but:

x_r	3.184	3.159834	4.00395 6	α: 3.183327	
x_r	3.183327	3.17584	3.37375 3	α: 3.183054	
x_r	3.183054	3.182409	3.19812 2	α: 3.183029	
x_r	3.183029	3.183024	3.18313 7	α: 3.183029	
				3.18303 to 5 dp	**[M1A1]**

[subtotal 8]

[TOTAL 24]

2

(i) Divided differences do not require data to be equally spaced (as ordinary differences do). Divided differences allow additional data to be added (unlike Lagrange). **[E1E1]**

[subtotal 2]

x	f
1	-3
2	-6.5
2.5	-8.03
3.5	-6.66
4	-2.25
4.5	5.65

				0.29333			
(ii)	1	-3	-3.5	3	1.064	-0.01911	**table**
				2.95333	1.00666		
	2	-6.5	-3.06	3	7		**[M1A1A1]**
				4.96666			
	2.5	-8.03	1.37	7			
	3.5	-6.66	8.82				
	4	-2.25					**estimates**
							[M1A1A1A1A1
	1.5	**-3**	**-4.75**	**-4.82333**	**-4.55733**	**-4.54778**	**]**
			linear	quadratic	cubic	quartic	

2nd dp unreliable (from data), 1st dp uncertain: could be -4.5 or -4.6 **[E1E1]**

[subtotal 10]

				4.96666	1.00666		
(iii)	3.5	-6.66	1.37	7	7	-4.4E-16	
				3.45666	1.00666		
	2.5	-8.03	3.853333	7	7		
	4	-2.25	2.125	5.47			**rearrange**
	2	-6.5	4.86				**data and**
	4.5	5.65					**re-run**
	3	**-6.66**	**-7.345**	**-8.58667**	**-8.335**	**-8.335**	**[M1A1]**
			linear	quadratic	cubic	quartic	

2nd dp unreliable (from data), 1st dp seems reliable: -8.3 **[E1]**

				1.00666		
4.5	5.65	15.8	6.98	7	-6.2E-16	
			4.96666	1.00666		
4	-2.25	8.82	7	7		
			2.95333			
3.5	-6.66	1.37	3			**rearrange**
2.5	-8.03	-3.06				**data and**
2	-6.5					**re-run**
5	**5.65**	**13.55**	**17.04**	**17.795**	**17.795**	**[M1A1]**
		linear	quadratic	cubic	quartic	

2nd dp unreliable (from data), 1st dp seems reliable: 17.8 **[E1]**
[subtotal 6]

(iv)	4	-2.25	15.8	6.98	1.006667	-4.4E-16	
	4.5	5.65	12.31	5.47	1.006667		
	3.5	-6.66	1.37	2.953333			**rearrange**
	2.5	-8.03	-3.06				**data and**
	2	-6.5					**re-run**
	4.16	**-2.25**	**0.278**	**-0.10171**	**-0.13786**	**-0.13786**	**[M1A1]**
	4.17	**-2.25**	**0.436**	**0.044422**	**0.006584**	**0.006584**	**[M1A1]**
	4.165	**5.65**	**1.52615**	**0.307571**	**-0.06582**	**-0.06582**	
	4.175	**-2.25**	**0.515**	**0.118012**	**0.079366**	**0.079366**	**[A1]**
			linear	quadratic	cubic	quartic	

Hence root is 4.17 to 2 dp **[A1]**
[subtotal 6]

[TOTAL 24]

3

(i)	Substitute central difference formulae for y' and y" to obtain given result (*)	**[M1A1]**
	Central difference formula for y' at x=0 to show $y_1 = y_{-1}$	**[M1A1]**
	Use of (*) to show $y_1 = (2h^2 - (1 + 2h)y_{-1})/(1 - 2h)$	**[M1A1]**
	Hence $y_1 = h^2$ as given	**[M1]**

h	**x**	**y**	**k**	
0.1	0	0	1	
	0.1	0.01		
	0.2	0.047618		**[M1A1]**
	0.3	0.124458		
	0.4	0.25785		
	0.5	0.473034		
	0.6	0.805379		
	0.7	1.301401		
	0.8	2.015508		
	0.9	2.996344		
	1	**4.253311**	**as required**	**[A1]**

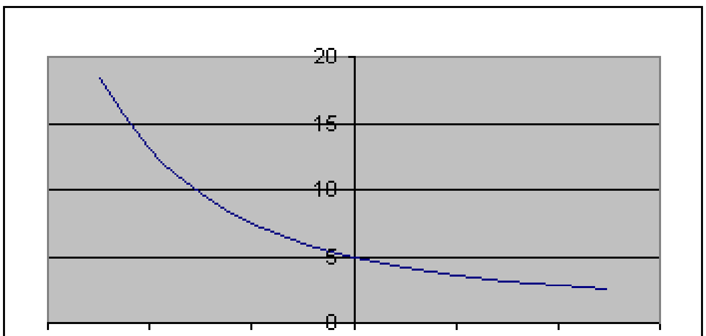
h	**β**	**diffs**	**ratio of diffs**	**extrapolated value**	
0.1	4.253311				
0.05	4.190790	-0.06252			**re-runs**
			0.24040 6		
0.025	4.175759	-0.01503			**[A1A1A1]**
0.012 5	4.172037	-0.00372	0.24760 4	4.17079 7	

ratio of differences approximately 0.25 so second order **[M1A1E1]**

4.17 to 2 dp is secure **[A1]**

[subtotal 17]

(ii)	**k**	**β**	
	-5	18.4	**mods**
	-4	13.1	**[M1A1]**
	-3	9.7	



-2	7.4	**values**
-1	6	**[A1A1A1]**
0	4.9	
1	4.2	**graph**
2	3.6	**[G2]**
3	3.2	
4	2.9	
5	2.6	

[subtotal 7]

[TOTAL 24]

4

(i) Diagonal dominance: modulus of diagonal element is greater than or equal to sum of moduli of other elements on the same row. **[E1]**

If diagonal dominance exists (with at least one inequality strict) convergence of Gauss-Seidel is assured. **[E1]**

G-S using the given non-dominant diagonal:

x	y	z	
			[M1]
0	0	0	
0.2	0.028571	-0.01587	**[M1A1]**
0.192381	0.041995	-0.0191	
0.186262	0.047335	-0.01866	
…	…	…	
0.180325	0.04918	-0.01639	
0.180327	0.04918	-0.01639	
0.180328	0.04918	-0.01639	
0.180328	0.04918	-0.01639	**[M1A1]**

[subtotal 7]

(ii)

a=3	**x**	**y**	**z**	**a=4**	**x**	**y**	**z**	
	0	0	0		0	0	0	**mods**
	0.333333	-0.06667	-0.04762		0.5	-0.25	-0.04167	**[M1A1]**
	0.447619	-0.12	-0.09116		0.9375	-0.64583	-0.07639	
	0.54449	-0.16267	-0.12987		1.583333	-1.25694	-0.10532	
	…	…	…		2.543403	-2.18808	-0.12944	**a=3**
	1	-0.33333	-0.33333		3.976273	-3.59684	-0.14953	**[M1A1]**
	1	-0.33333	-0.33333		6.119551	-5.72002	-0.16628	
	1	-0.33333	-0.33333		9.329443	-8.91317	-0.18023	**a=4**
					14.1401	-13.7099	-0.19186	**[M1A1]**
					21.35259	-20.9107	-0.20155	
					…	…	…	
					2.6E+18	-2.6E+18	0	
					3.91E+18	-3.9E+18	0	

5.86E+18	-5.9E+18	0

G-S scheme converges for a=3.3 **[M1A1]**
diverges for a=3.4 **[M1A1]**
(diverges for a=3.35)
So a=3.3 (to 1dp) is required value **[A1]**

[subtotal 11]

(iii) Gauss-Jacobi

a=0	**x**	**y**	**z**	
	0	0	0	
	0.166667	0.125	0.1	**[M1A1]**
	0.054167	-0.00833	-0.04583	
	0.19375	0.120833	0.077083	
	0.067708	-0.01042	-0.05729	
	0.200521	0.119792	0.071354	**[M1A1]**
	…	…	…	
	0.202778	0.119444	0.069444	
	0.072222	-0.01111	-0.06111	
	0.202778	0.119444	0.069444	
	0.072222	-0.01111	-0.06111	
	0.202778	0.119444	0.069444	
	0.072222	-0.01111	-0.06111	**[A1]**

Diverges: diagonal dominance not strict. **[E1]**

[subtotal 6]

[TOTAL 24]

4777 - Numerical Computation

General Comments

The candidature for this paper was, once again, small. Standards varied greatly; the best scored full marks, but at the other extreme there was little knowledge of the necessary theory and little familiarity with the techniques. Candidates all seemed at home in the use of a spreadsheet, however.

Comments on Individual Questions

Section A

1) **Solution of an equation; acceleration**
This question was frequently well done. In part (i) the algebra was often convoluted and sometimes unconvincing. Parts (ii) and (iii) were generally correct, though a few candidates did not appreciate that, when Aitken's acceleration formula is used to produce an improved estimate, that value should be used to re-start the process.

2) **Divided differences**
Parts (i) and (ii) were generally done correctly with candidates showing a good grasp of theory and practice. In part (iii), a common error was failing to re-order the data points relative to the point at which the function is to be evaluated. In part (iv) it was often unclear what candidates were doing. Using trial and error it is possible to show that the function, estimated by a quartic, changes sign between x = 4.165 and 4.175. Hence the root is estimated as 4.17 to 2 d.p.

3) **Second order differential equation**
This was the least popular question, but those who tackled it mostly scored highly. The algebra in part (i) was done well, as were the solution and the graph in parts (ii) and (iii).

4) **Iterative solution of a system of linear equations**
This, too, was a question on which most candidates scored high marks. The level of understanding of the methods was good, though some confused the Gauss-Seidel and Gauss-Jacobi versions. Those who worked with the equations as separate entities fared a little better than those who used decomposition on the coefficient matrix.