OCR
RECOGNISING ACHIEVEMENT

ADVANCED GCE

MATHEMATICS (MEI) 4777

Numerical Computation

Tuesday 23 June 2009
Morning

Duration: 2 hours 30 minutes

Candidates answer on the Answer Booklet

OCR Supplied Materials:
- 8 page Answer Booklet
- MEI Examination Formulae and Tables (MF2)
- Graph paper

Other Materials Required:
None

INSTRUCTIONS TO CANDIDATES

- Write your name clearly in capital letters, your Centre Number and Candidate Number in the spaces provided on the Answer Booklet.
- Use black ink. Pencil may be used for graphs and diagrams only.
- Read each question carefully and make sure that you know what you have to do before starting your answer.
- Answer any **three** questions.
- Do **not** write in the bar codes.
- Additional sheets, including computer print-outs, should be fastened securely to the Answer Booklet.

COMPUTING RESOURCES

- Candidates will require access to a computer with a spreadsheet program and suitable printing facilities throughout the examination.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets **[ ]** at the end of each question or part question.
- In each of the questions you are required to write spreadsheet routines to carry out various numerical analysis processes.
- You will not receive credit for using any numerical analysis functions which are provided within the spreadsheet. For example, many spreadsheets provide a solver routine; you will not receive credit for using this routine when asked to write your own procedure for solving an equation.
  You may use the following built-in mathematical functions: square root, sin, cos, tan, arcsin, arccos, arctan, ln, exp.
- For each question you attempt, you should submit print-outs showing the spreadsheet routine you have written and the output it generates. It will be necessary to print out the formulae in the cells as well as the values in the cells.
  You are not expected to print out and submit everything your routine produces, but you are required to submit sufficient evidence to convince the examiner that a correct procedure has been used.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.
- The total number of marks for this paper is **72**.
- This document consists of **4** pages. Any blank pages are indicated.

1 **(i)** The equation $x = \mathrm{g}(x)$ has a root $x = \alpha$. State a condition on the derivative of $\mathrm{g}(x)$ that will ensure convergence of the iteration $x_{r+1} = \mathrm{g}(x_r)$ provided x_0 is close enough to α.

Obtain the relaxed iteration $x_{r+1} = \lambda\,\mathrm{g}(x_r) + (1 - \lambda)\,x_r$. Show that, for fastest convergence,

$$\lambda = \frac{1}{1 - \mathrm{g}'(\alpha)}.$$

State how a value for λ would be chosen in practice. **[7]**

(ii) Use a spreadsheet to show graphically that the equation

$$x = 3\sin x - 0.5$$

(where x is in radians) has two roots in the interval (0, 3). Use your graph to give approximate values for these roots.

Show that the iteration

$$x_{r+1} = 3\sin x_r - 0.5$$

does not converge to either root. You should try several values of x_0 in each case.

Use the method of relaxation to find each root correct to 6 decimal places. **[17]**

2 The Gaussian 3-point integration formula has the form

$$\int_{-h}^{h} \mathrm{f}(x)\,\mathrm{d}x = a\,\mathrm{f}(-\alpha) + b\,\mathrm{f}(0) + a\,\mathrm{f}(\alpha).$$

(i) By considering $\mathrm{f}(x) = 1, x, x^2, x^3, x^4$, obtain the three equations that determine a, b and α. Verify that these equations are satisfied by

$$\alpha = \sqrt{\tfrac{3}{5}}\,h,$$

$$a = \tfrac{5}{9}\,h,$$

$$b = \tfrac{8}{9}\,h.$$ **[8]**

(ii) Taking $h = \frac{\pi}{8}$ initially, use the Gaussian 3-point rule to estimate the value of

$$\int_0^{\frac{\pi}{4}} \sqrt{1 + \tan x}\,\mathrm{d}x\,.$$

Repeat the process, halving h as necessary, in order to establish the value of the integral correct to 6 decimal places. **[12]**

(iii) Determine, correct to 3 decimal places, the value of k such that

$$\int_0^{\frac{\pi}{4}} \sqrt{1 + k\tan x}\,\mathrm{d}x = 1.$$ **[4]**

3 The second order differential equation

$$\frac{d^2y}{dx^2} + \sqrt{x}\,\frac{dy}{dx} + xy = 1$$

with initial conditions $x = 0$, $y = 0$, $\frac{dy}{dx} = a$, is to be solved for various values of a using finite difference methods.

(i) Consider first the case $a = 1$.

Show that, in the usual notation,

$$y_{r+1} = \frac{2(2 - h^2 x_r)\,y_r + (h\sqrt{x_r} - 2)\,y_{r-1} + 2h^2}{2 + h\sqrt{x_r}},$$

and that

$$y_1 = h + \tfrac{1}{2}h^2. \qquad (*)$$ **[8]**

(ii) Obtain a solution from $x = 0$ to $x = 5$ with $h = 0.1$. Use your spreadsheet to produce a graph of this solution. **[9]**

(iii) Modify (*) to allow different values of a to be used.

Still using $h = 0.1$, find, correct to 1 decimal place, a negative value of a for which the graph of the solution curve crosses the axis very close to $x = 2$. **[7]**

4 The system of linear equations with augmented matrix

$$\left(\begin{array}{cccc|c} a & 1 & b & 1 & 1 \\ 1 & a & 1 & b & 0 \\ b & 1 & a & 1 & 0 \\ 1 & b & 1 & a & 0 \end{array}\right)$$

is to be solved, using the Gauss-Seidel method, for various values of a and b.

(i) Explain the condition of diagonal dominance. State a condition on a and b that will ensure convergence. **[3]**

(ii) Set up a spreadsheet implementing the Gauss-Seidel method and allowing the user to vary the values of a and b.

Show that convergence does occur in the case $a = 4$, $b = 2$, and does not occur in the case $a = 2$, $b = 4$. **[12]**

(iii) Investigate the case $a = 2$, $b = 0$. What do your results indicate about diagonal dominance? **[4]**

(iv) By modifying your spreadsheet find the inverse of the following matrix.

$$\begin{pmatrix} 4 & 1 & 2 & 1 \\ 1 & 4 & 1 & 2 \\ 2 & 1 & 4 & 1 \\ 1 & 2 & 1 & 4 \end{pmatrix}$$

[5]

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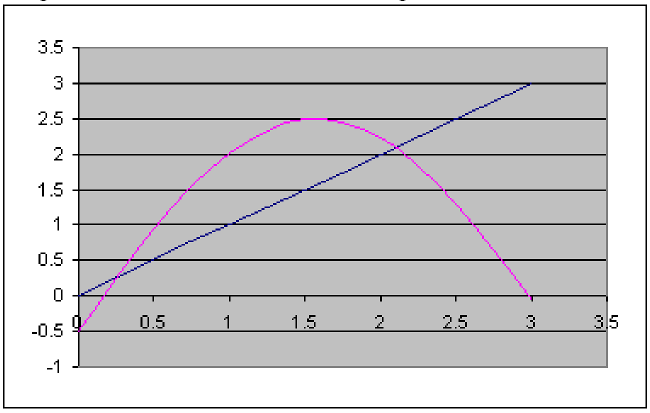
1(i) -1 < g'(α) < 1 **[B1]**

E.g. Multiply both sides of $x = g(x)$ by λ and add $(1 - \lambda)x$ to both sides. **[M1A1]**
Derivative of rhs set to zero at root: $\lambda g'(\alpha) + 1 - \lambda = 0$ **[M1A1]**
algebra to obtain given result **[A1]**
In practice use an initial estimate x_0 in place of α **[A1]**

[subtotal 7]

(iii)



[G3]

Roots approximately 0.25, 2.1 **[B1B1]**

Eg:

r	x_r	x_r	x_r	x_r	x_r	x_r
0	0	0.2	0.4	2	2.2	2.4
1	-0.5	0.096008	0.668255	2.227892	1.925489	1.52639
2	-1.93828	-0.21242	1.358852	1.875308	2.31326	2.497043
3	-3.29971	-1.13247	2.432871	2.36198	1.710416	1.302517
4	-0.02763	-3.21639	1.452591	1.609012	2.470807	2.392685
5	-0.58289	-0.2758	2.479066	2.49781	1.364805	1.542517
6	-2.15131	-1.31696	1.345334	1.300676	2.436576	2.498801
7	-3.00855	-3.40387	2.424072	2.391217	1.444139	1.298298
8	-0.89795	0.277847	1.472555	1.545741	2.475969	2.389305
9	-2.84615	0.322857	2.485535	2.499058	1.352649	1.549934
10	-1.37349	0.451832	1.329994	1.297679	2.4289	2.499347

No convergence in each case **[M1A1A1]**

Let $g(x) = 3 \sin x - 0.5$
Then $g'(x) = 3 \cos x$
So $\lambda = 1 / (1 - 3 \cos\alpha)$ **[M1A1]**

Smaller root: $\lambda =$ -0.52446 (approx -0.5) Larger root: $\lambda =$ 0.397687 (approx 0.4) **[M1A1A1]**

r	x_r
0	0.25
1	0.253894
2	0.254078

NB: must be using relaxatio

r	x_r
0	2.1
1	2.095851
2	2.095866

	n				
3	0.254087		3	2.095866	
4	0.254088		4	2.095866	**[M1A1]**
5	0.254088		5	2.095866	**[M1A1]**

[subtotal 17]
[TOTAL 24]

2(i) f(x) = 1 $2h = 2a + b$ **[M1A1]**
f(x) = x, x^3 give 0 = 0 **[M1A1]**
f(x) = x^2 $2h^3/3 = 2a\alpha^2$ **[A1]**
f(x) = x^4 $2h^5/5 = 2a\alpha^4$ **[A1]**
Convincing algebra to verify given results **[A1A1]**
[subtotal 8]

(ii)

	L	R	*m*	*h*	×1	×2		
	0	0.785398	0.392699	0.392699	0.088516	0.696882		
function values			1.189207		1.043431	1.35535		**setup:**
weights			0.349066		0.218166	0.218166		**[M3A3]**
integral			0.415112		0.227641	0.295691	**0.938444**	**[A1]**

	L	R	m	h	×1	×2		
	0	0.392699	0.19635	0.19635	0.044258	0.348441		
function values			1.094949		1.021903	1.167589		
weights			0.174533		0.109083	0.109083		
integral			0.191105		0.111472	0.127364	0.429941	
	0.392699	0.785398	0.589049	0.19635	0.436957	0.74114		
function values			1.29158		1.211226	1.383901		**repeat: [M2]**
weights			0.174533		0.109083	0.109083		
integral			0.225423		0.132124	0.15096	0.508508	
							0.938449	**[A1]**

Either repeat with h halved to verify that 0.938449 is correct to 6 dp **[M1A1]**
Or observe that the method is converging so rapidly that 0.938449 will be correct to 6dp **or [E1A1]**
[subtotal 12]

(iii) Use routine known to deliver 6dp and vary k:

	L	R	m	h	×1	×2		
						k =	1.46572	
	0	0.392699	0.19635	0.19635	0.044258	0.348441		
function values			1.136464		1.031946	1.237918		
weights			0.174533		0.109083	0.109083		
integral			0.19835		0.112568	0.135036		0.445954
	0.392699	0.785398	0.589049	0.19635	0.436957	0.74114		**modify**
function values			1.406898		1.297918	1.530164		**[M1A1]**
weights			0.174533		0.109083	0.109083		
integral			0.24555		0.141581	0.166915	0.554046	
							1.000000	

k	1.465	1.466	1.467	
integral	0.999908	1.000036	1.000163	**find k [M1A1]**

hence k = 1.466

[subtotal 4]
[TOTAL 24]

3(i) Use central difference formulae for 2nd and 1st derivatives to obtain first given result **[M1A1A1]**

Hence obtain $y_1 = h^2 - y_{-1}$ **[M1A1]**

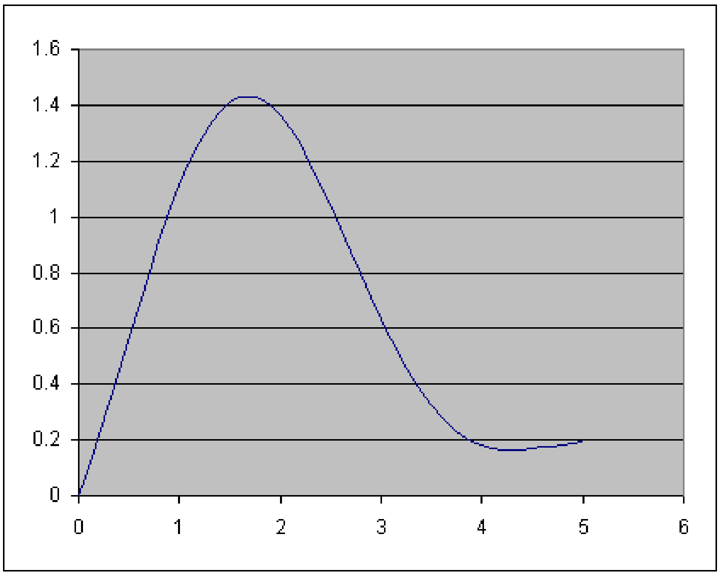
Use central difference to obtain $y_1 - y_{-1} = 2h$ **[M1A1]**

Hence given result for y_1 **[M1]**

[subtotal 8]

(ii)

h	x	y
0.1	0	0
	0.1	0.105
	0.2	0.216472
	0.3	0.332426
	0.4	0.450961
	0.5	0.570174
	0.6	0.68815
	0.7	0.802981
	0.8	0.912793
	0.9	1.015786
	1	1.11027
	1.1	1.194705
	1.2	1.26774
	1.3	1.328248
	1.4	1.375354
	1.5	1.40846
	1.6	1.42726
	1.7	1.431751
	1.8	1.42223
	1.9	1.399287
	2	1.363785
	2.1	1.316838
	2.2	1.259773
	2.3	1.194096
	2.4	1.121445
	2.5	1.04354
	2.6	0.962141
	2.7	0.878993
	2.8	0.79578
	2.9	0.714082
	3	0.635337
	3.1	0.560807
	3.2	0.491549
	3.3	0.428404
	3.4	0.371982
	3.5	0.322662
	3.6	0.280597
	3.7	0.245729
	3.8	0.217808
	3.9	0.196416
	4	0.180999
	4.1	0.170894
	4.2	0.165365
	4.3	0.163635
	4.4	0.164915
	4.5	0.168435
	4.6	0.173469

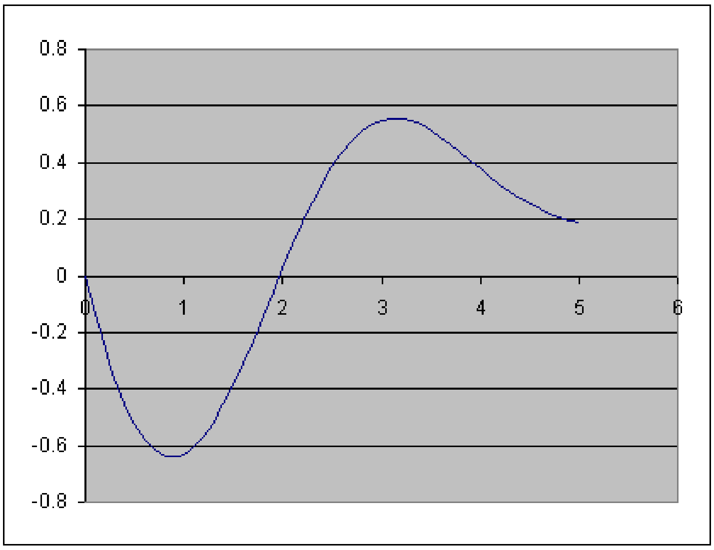


4.7	0.179352
4.8	0.185502
4.9	0.191424
5	0.196725

setup	numbers	graph
[M3]	[A3]	[A3]
		[subtotal 9]

(ii) Obtain formula $y_1 = ah + 0.5h^2$ **[M1A1]**
Modify routine **[M1A1]**
Trial on a to obtain a = -1.4 or -1.5 **[M1A1G1]**

h	*x*	*y*
0.1	0	0
a	0.1	-0.135
-1.4	0.2	-0.25582
	0.3	-0.36107
	0.4	-0.44993
	0.5	-0.5219
	0.6	-0.57677
	0.7	-0.6146
	0.8	-0.63565
	0.9	-0.64047
	1	-0.6298
	1.1	-0.60462
	1.2	-0.56614
	1.3	-0.51572
	1.4	-0.45494
	1.5	-0.3855
	1.6	-0.3092
	1.7	-0.22792
	1.8	-0.14356
	1.9	-0.05802
	2	0.026884
	2.1	0.109408
	2.2	0.187962
	2.3	0.26113
	2.4	0.327696
	2.5	0.386672
	2.6	0.437316
	2.7	0.479135
	2.8	0.51189
	2.9	0.535589
	3	0.550471
	3.1	0.556986
	3.2	0.555768
	3.3	0.547604
	3.4	0.533401
	3.5	0.514147
	3.6	0.490876
	3.7	0.464631
	3.8	0.43643
	3.9	0.40724
	4	0.377942
	4.1	0.349319
	4.2	0.322033



4.3	0.296623
4.4	0.27349
4.5	0.252909
4.6	0.235026
4.7	0.219875
4.8	0.207386
4.9	0.197404
5	0.189706

[subtotal 7]
[TOTAL24]

4(i) Diagonal dominance: the magnitude of the diagonal element in any row is greater than or equal to the sum of the magnitudes of the other element. **[E1]**
| *a* | > | *b* | + 2 will ensure convergence. (> required as dominance has to be strict) **[E1E1]**
[subtotal 3]

(ii)

					a	b
4	1	2	1	1	4	2
1	4	1	2	0		
2	1	4	1	0		
1	2	1	4	0		

0	0	0	0	
0.25	-0.0625	-0.10938	-0.00391	**setup**
0.321289	-0.05103	-0.14691	-0.01808	**[M3A3]**
0.340733	-0.03941	-0.15599	-0.02648	
0.344469	-0.03388	-0.15715	-0.02989	
0.344515	-0.0319	-0.15681	-0.03098	
0.344124	-0.03134	-0.15648	-0.03124	
0.343886	-0.03123	-0.15633	-0.03127	
0.343789	-0.03123	-0.15627	-0.03127	
0.343758	-0.03124	-0.15625	-0.03126	
0.34375	-0.03125	-0.15625	-0.03125	
0.343749	-0.03125	-0.15625	-0.03125	
0.34375	-0.03125	-0.15625	-0.03125	**values**
0.34375	-0.03125	-0.15625	-0.03125	**[A3]**

					a	*b*
2	1	4	1	1	2	4
1	2	1	4	0		
4	1	2	1	0		
1	4	1	2	0		

0	0	0	0	
0.5	-0.25	-0.875	0.6875	
2.03125	-1.95313	-3.42969	4.605469	
6.033203	-10.5127	-9.11279	22.56519	
12.69934	-46.9236	-13.2195	94.10735	
3.347054	-183.278	37.89147	345.9377	
-156.613	-632.515	456.5137	1115.079	**values**
-1153.81	-1881.51	2690.835	2994.509	**[A3]**
-5937.67	-4365.6	12560.88	5419.593	

[subtotal 12]

(iii) No convergence when $a = 2$, $b = 0$ **[M1A1]**
Indicates that non-strict diagonal dominance is not sufficient **[E1E1]**

[subtotal 4]

(iv) Use RHSs 1,0,0,0 0,1,0,0 0,0,1,0 0,0,0,1 to obtain inverse as **[M1]**

0.34375	-0.03125	-0.15625	-0.03125	**[A1]**
-0.03125	0.34375	-0.03125	-0.15625	**[A1]**
-0.15625	-0.03125	0.34375	-0.03125	**[A1]**
-0.03125	-0.15625	-0.03125	0.34375	**[A1]**

[subtotal 5]
[TOTAL 24]

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General Comments

Once again, the candidature for this paper was small. However, candidates were mostly well prepared and there were some high scores.

Comments on Individual Questions

1) **(Solution of an equation; relaxation)**

   The algebra to set up relaxation was not generally well handled, but the numerical solution in part (ii) was better.

2) **(Gaussian integration)**

   By contrast with question 1, the algebra to set up the Gaussian 3-point rule was handled well. In the numerical part, some candidates struggled with the fact that the required integral is not centred on zero.

3) **(Second order differential equation; finite difference method)**

   The algebra at the start of the question proved tricky for some, but the numerical work was carried out successfully.

4) **(Gauss-Seidel and Gauss-Jacobi methods)**

   Those candidates who attempted the question knew what to do and scored highly on it.