OCR
RECOGNISING ACHIEVEMENT

OXFORD CAMBRIDGE AND RSA EXAMINATIONS

Advanced Subsidiary General Certificate of Education
Advanced General Certificate of Education

MEI STRUCTURED MATHEMATICS 4776

Numerical Methods

Wednesday **25 MAY 2005** Afternoon 1 hour 30 minutes

Additional materials:
Answer booklet
Graph paper
MEI Examination Formulae and Tables (MF2)

TIME 1 hour 30 minutes

INSTRUCTIONS TO CANDIDATES

- Write your name, centre number and candidate number in the spaces provided on the answer booklet.
- Answer **all** the questions.
- You are permitted to use a graphical calculator in this paper.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets [ ] at the end of each question or part question.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.
- Final answers should be given to a degree of accuracy appropriate to the context.
- The total number of marks for this paper is 72.

This question paper consists of 5 printed pages and 3 blank pages.

Section A (36 marks)

1 In the table below the values of x are exact and the values of the function $f(x)$ are correct to 4 decimal places.

x	2	2.1	2.2	2.3	2.4
$f(x)$	4.0000	4.2871	4.5948	4.9246	5.2780

(i) Obtain an estimate of the gradient at $x = 2$ using the forward difference formula with $h = 0.4$.

Find two further estimates by successively halving h. [4]

(ii) By considering how the differences between these three estimates reduce, obtain a better estimate, giving your answer to an appropriate degree of accuracy. [4]

2 (i) Explain by means of a simple example the difference between rounding and chopping when numbers are represented in a calculator or computer. [1]

(ii) A cheap calculator stores numbers in the range $(0, 1)$ in decimal form rounded to 7 decimal places.

The number $\frac{2}{3}$ is stored in decimal form. Write down the absolute error in the stored value.

Write down the maximum possible error in the stored value of x, where $0.1 < x < 1$. Find the maximum possible relative error in the stored value of x. [6]

3 Show that the equation

$$x^5 - 3x - 2 = 0$$

has a root, α, in the interval $1.4 < x < 1.5$.

Use the secant method with $x_0 = 1.4$ and $x_1 = 1.5$ to find x_2, x_3, x_4.

Give a value for α to the accuracy that is appropriate. [8]

4 Obtain estimates of the value of

$$I = \int_2^4 \sqrt{(1+\sqrt{x})}\,dx$$

using the mid-point rule and the trapezium rule, each with $h = 2$. Give your answers to 6 decimal places.

Hence obtain a Simpson's rule estimate of I.

You are now given that the Simpson's rule estimates of I obtained as above but starting with $h = 1$ and $h = 0.5$ are 3.299 231 and 3.299 238. By considering the differences in the three Simpson's rule estimates, obtain the best estimate you can. Give your answer to an appropriate number of significant figures. [8]

5 The series

$$1 + \left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{3}\right)^2 + \ldots + \left(\tfrac{1}{10000}\right)^2$$

is being summed in a computer program that works to a precision of 7 significant figures.

When the terms are summed in the order shown, the value obtained is 1.644 725. When the same terms are summed in reverse order, the value obtained is 1.644 834.

Explain why computations of this type do not give exact answers.

Explain further why the two approaches used give different answers. State which of the answers is likely to be more accurate and explain why. [5]

Section B (36 marks)

6 **(i)** Copy and complete the following difference table as far as third differences.

x	$\mathrm{f}(x)$	$\Delta \mathrm{f}(x)$	$\Delta^2 \mathrm{f}(x)$	$\Delta^3 \mathrm{f}(x)$
1	4			
		-3		
2	1		·	
		3		·
3	4		·	
		$a-4$		·
4	a		·	
		$76-a$		
5	76			

[4]

(ii) Given that $\mathrm{f}(x)$ is a cubic polynomial, show that $a = 25$.

Use Newton's forward difference interpolation formula to obtain $\mathrm{f}(x)$. [8]

(iii) Find, correct to 3 significant figures, the minimum value of $\mathrm{f}(x)$ for $1 < x < 5$. [3]

(iv) The cubic polynomial $\mathrm{f}(x)$ could have been found using only the following data.

x	1	2	3	5
$\mathrm{f}(x)$	4	1	4	76

Write down, *but do not simplify*, an expression for $\mathrm{f}(x)$ based on these data. [3]

7 **(i)** Sketch, on the same axes, the graphs with equations

$$y = 3 - x,$$

$$y = 4\cos x,$$

where x is in radians and $0 \leqslant x \leqslant 2\pi$.

Hence show that the equation

$$3 - x = 4\cos x \qquad (*)$$

has two roots, α and β with $\alpha < \beta$, in the interval $[0, 2\pi]$. [3]

(ii) Use the iteration

$$x_{r+1} = \arccos\left(\tfrac{1}{4}(3 - x_r)\right)$$

to find the root α correct to 3 decimal places. [4]

(iii) By considering the ratio of the differences between the values of x_r, show that the iteration in part **(ii)** displays first order convergence. [3]

(iv) Given that the derivative of $\cos x$ is $(-\sin x)$ when x is in radians, obtain the following Newton-Raphson iteration for equation (*).

$$x_{r+1} = x_r + \frac{3 - x_r - 4\cos x_r}{1 - 4\sin x_r}.$$

Use this iteration to find β correct to 4 decimal places. [5]

(v) By considering the ratio of the differences between the values of x_r, show that the iteration in part **(iv)** displays convergence that is faster than first order. [3]

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1(i)	h	0.4	0.2	0.1			
	est f '(2)	3.195	2.974	2.871		**M1A1A1A1**	**[4]**
(ii)	differences		-0.221	-0.103		**M1A1**	

differences approximately halving so extrapolate to 2.871 - 0.103 = 2.768.
Last figure unreliable so 2.77. Accept argument to 2.8. **M1A1** **[4]**

2(i)	E.g. 2/3 rounded to 0.666 666 7, chopped to 0.666 666 6		**E1**	**[1]**
(ii)	2/3 stored as 0.666 666 7	Absolute error 0.000 000 033...	**A1A1**	
	mpe is 0.000 000 05		**A1**	
	mpre is greatest when x is least		**M1**	
	mpre is 0.000 000 05 / 0.1 =	5 * 10^ -7 5E-07	**M1A1**	**[6]**

3	x	f(x)				
	1.4	-0.82176				
	1.5	1.09375	root in the interval (1.4, 1.5)		**B1**	
		r	Xr	f(Xr)		
		0	1.4	-0.82176		
		1	1.5	1.09375	**M1**	
		2	1.4429	-0.07436	**M1A1**	
		3	1.446535	-0.00609	**A1**	
		4	1.446859	3.88E-05	**M1A1**	
	Root at 1.447 seems secure.				**B1**	**[8]**

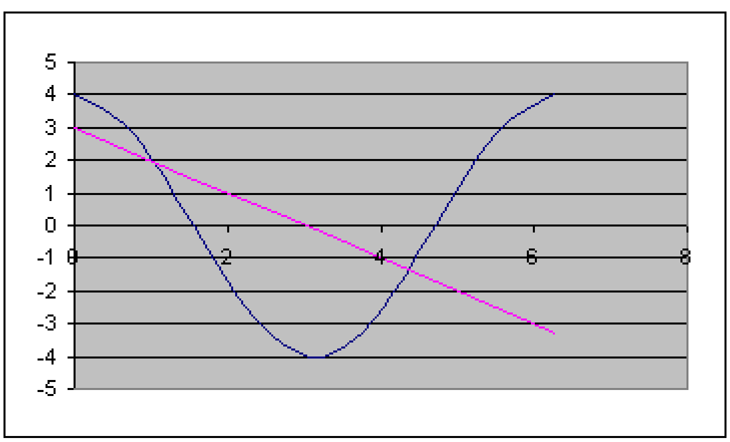
4	x	f(x)						
	2	1.553774		M =		3.305783	**M1A1**	
	3	1.652892		T =		3.285825	**A1**	
	4	1.732051		S =	(2*M + T) / 3 =	3.299130	**M1A1**	
	S(h=2)	3.299130	diffs					
	S(h=1)	3.299231	0.0001					
	S(h=0.5)	3.299238	7 E -06					
	Differences reducing very rapidly. 3.29924 seems secure.						**M1A1A1**	**[8]**

5	Computations of this type contain rounding errors	**E1**
	The rounding errors will be different when the two sums are computed	**E1**
	Adding from large to small loses precision (the small number is lost)	**E1**
	Adding from small to large allows each number to contribute to the sum	**E1**
	Hence the second sum is likely to be more accurate	**E1**

	x	f(x)	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$		
6(i)	1	4					
			–3				
	2	1		**6**			
			3		**a – 13**		
	3	4		**a – 7**			
			a – 4		**87 – 3a**		
	4	a		**80 – 2a**		**A4**	
			76 – a			**(-1 each**	
	5	76				**error)**	**[4]**

(ii)	87 – 3a = a – 13 gives a = 25		**M1**	
	$f(x) = 4 - 3(x-1) + 6(x-1)(x-2)/2 + 12(x-1)(x-2)(x-3)/6$		**M1A1A1A1A1**	
	$= 4 - 3x + 3 + 3x^2 - 9x + 6 + 2x^3 - 12x^2 + 22x - 12$		**A1**	
	$= 2x^3 - 9x^2 + 10x + 1$		**A1**	**[8]**
(iii)	$f'(x) = 6x^2 - 18x + 10 = 0$	*Algebra may appear in (iii) rather than (ii) for full credit*	**M1**	
	x = 2.26 (2.26376)		**A1**	
	f(2.26...) = 0.718		**A1**	**[3]**
(iv)	$f(x) = 4\,(x - 2)(x - 3)(x - 5)/(1 - 2)(1 - 3)(1 - 5)$ + three similar terms		**M1A1A1**	**[3]**
				Total 18

7(i)



	G1	
	G1	
shows two roots	**E1**	**[3]**

(ii) E.g.:

r	0	1	2	3	4	5	6	7
Xr	1	1.04720	1.06077	1.06465	1.06576	1.06608	1.06617	1.06620

M1A1A1

alpha = 1.066 correct to 3 decimal places **A1** **[4]**

(iii)

0	1	2	3	4	5	6	
1	1.04720	1.06077	1.06465	1.06576	1.06608	1.06617	
diffs	0.04720	0.01357	0.00388	0.00111	0.00032	0.00009	
ratio of diffs		0.28756	0.28615	0.28575	0.28564	0.28561	**M1A1**

ratios (approx) constant so first order convergence. **E1** **[3]**

(iv) Obtain N-R iteration (beware printed answer) **M1A1**

E.g.:

r	0	1	2	3	4	
Xr	5	4.35177	4.36435	4.36432	4.36432	**M1A1**

beta = 4.3643 correct to 4 decimal places **A1** **[5]**

(v)

0	1	2	3	4	
5	4.35177	4.36435	4.36432	4.36432	
diffs	-0.64823	0.01258	-0.00002	0.00000	
ratio of diffs		-0.01940	-0.00184	0.00000	**M1A1**

ratios getting (much) smaller so faster than first order **E1** **[3]**

Total 18

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General Comments

The increase in numbers taking this paper was welcome. Most were reasonably well prepared, though as usual conceptual understanding was not as strong as arithmetical facility.

Comments on Individual Questions

1) **Numerical differentiation**

This question proved accessible to almost everyone. The extrapolation in part (ii) was generally well done, though judging the appropriate level of accuracy was found more difficult.

2) **Errors and accuracy**

Part (i) on rounding and chopping was very easy. In part (ii), finding the maximum possible relative error proved more difficult. In some cases answers were given with no explanation or with an explanation that was difficult to follow.

3) **Secant method to solve an equation**

This question was generally well done, though some candidates did not use the method specified. There can be no credit for using an alternative method even if it gives the correct numerical solution.

4) **Numerical integration**

The numerical work was well done, though a surprising number of candidates did not give answers to the required precision. The extrapolation defeated some, but for many it proved no problem.

5) **Errors in summing a series**

This question was worth 5 marks, but many answers made only one or two points. The question contains several quite distinct requests and, as a matter of examination technique, candidates would be advised to respond carefully to each one in turn.

6) **Difference table, Newton's forward difference method**

The missing values in the difference table were found correctly by most, though some made sign errors. Demonstrating the value of *a* presented little difficulty. The algebra required to obtain the cubic was more of a challenge, however, and there were many errors. In part (iii), a significant number did not think to find the minimum by differentiation. In part (iv), a number of candidates did not recognize the need to use Lagrange's formula. Those who did sometimes confused the *x* and f(*x*) values.

7) **Fixed point iteration and the Newton-Raphson method**

This was found to be a challenging question. In part (i) the graphs defeated some, while others drew correct graphs but said nothing about the roots. Part (ii) was generally successful, but in part (iii) a good many seemed to think that the ratio of differences should have been 0.25. The algebra in part (iv) was difficult for some and there were many dubious manipulations of signs. Part (iv) was too much for many. All that is required here is to show that the ratio of differences decreases substantially.