

**OXFORD CAMBRIDGE AND RSA EXAMINATIONS**

**Advanced Subsidiary General Certificate of Education  
Advanced General Certificate of Education**

**MEI STRUCTURED MATHEMATICS**

**4776**

Numerical Methods

Wednesday      **25 JANUARY 2006**      Morning      1 hour 30 minutes

Additional materials:  
8 page answer booklet  
Graph paper  
MEI Examination Formulae and Tables (MF2)

**TIME**      1 hour 30 minutes

**INSTRUCTIONS TO CANDIDATES**

- Write your name, centre number and candidate number in the spaces provided on the answer booklet.
- Answer **all** the questions.
- You are permitted to use a graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.

**INFORMATION FOR CANDIDATES**

- The number of marks is given in brackets [ ] at the end of each question or part question.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.
- The total number of marks for this paper is 72.

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**This question paper consists of 4 printed pages.**

## Section A (36 marks)

1 Show that if

$$X = x(1 + r)$$

then

$$\frac{1}{X} \approx \frac{1}{x}(1 - r)$$

for sufficiently small  $r$ .

Explain carefully what this means in terms of relative errors. Illustrate your answer in the case  $X = 10$ ,  $x = 9.8$ . [6]

2 The equation

$$\frac{1}{x} = \sin x \quad (*)$$

(where  $x$  is in radians) has two roots,  $\alpha$  and  $\beta$ , in the interval  $0 < x < \pi$ .

(i) Given that  $\alpha \approx 1$ , use an iteration based on a simple rearrangement of (\*) to find  $\alpha$  correct to 3 significant figures. [4]

(ii) Verify that  $\beta = 2.773$  is correct to 3 decimal places. [3]

3 The integral  $I = \int_1^3 f(x) dx$  is being evaluated numerically. The following values have been obtained.

$h$	mid-point rule	trapezium rule
2	2.602 42	2.448 66
1	2.569 82	

Obtain 2 further trapezium rule estimates of  $I$ .

Obtain 2 Simpson's rule estimates of  $I$ .

Give a value for  $I$  to the accuracy that appears justified. [7]

- 4 The function  $f(x)$  has the values shown in the table.

$x$	1	2	3	4	5
$f(x)$	3.0	4.5	5.4	6.2	6.7

- (i) Obtain two estimates of the gradient  $f'(3)$  using the central difference method. [3]
- (ii) Given that the values of  $x$  are exact but that the values of  $f(x)$  are rounded to 1 decimal place, determine a range of possible values of  $f'(3)$  from each estimate. Comment on your results.

Give a value for  $f'(3)$  to the accuracy that appears justified. [5]

- 5 The function  $g(x)$  is quadratic. The following values are known.

$x$	1	3	4
$g(x)$	4	1	11

Use Lagrange's method to determine  $g(2)$ .

Check your answer by drawing up a difference table for  $g(x)$ . [8]

### Section B (36 marks)

- 6 (i) Show that the curve  $y = x^{10} - 10x + 1$  has exactly one turning point.

Show that the equation

$$x^{10} - 10x + 1 = 0 \quad (*)$$

has exactly two real roots and that these roots both lie in the interval  $[0, 2]$ . [7]

- (ii) Use the Newton-Raphson method to find the larger root correct to 4 decimal places. [6]
- (iii) Obtain another iteration based on a rearrangement of (\*) and hence, *without using a calculator*, show that the smaller root is almost exactly  $0.1 + 0.1^{11}$ . [5]

7 The number  $e$  satisfies the relationship  $\log_e N = \int_1^N \frac{1}{x} dx$ . Hence it follows, for example, that  $\log_e 5 = \int_1^5 \frac{1}{x} dx$ .

(i) Use the mid-point rule to show that  $\log_e 5 \approx \frac{1}{1.5} + \frac{1}{2.5} + \frac{1}{3.5} + \frac{1}{4.5}$ . (\*)

Given that, correct to 8 decimal places,  $\log_e 5 = 1.609\ 437\ 91$ , find the error in the approximation (\*). [5]

(ii) Correct to 8 decimal places,  $\log_e 2 = 0.693\ 147\ 18$ . Hence or otherwise obtain the values of  $\log_e 10$ ,  $\log_e 20$ ,  $\log_e 40$  and  $\log_e 80$ , giving your answers correct to 6 decimal places. [3]

It is known that, as  $N$  increases, the value of the expression

$$\log_e N - \left( \frac{1}{1.5} + \frac{1}{2.5} + \frac{1}{3.5} + \dots + \frac{1}{(N-0.5)} \right)$$

tends to a constant  $k$ .

(iii) Use the information in the table to obtain 4 estimates,  $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ , of  $k$ . Give these estimates to 6 decimal places.

$N$	10	20	40	80
$\frac{1}{1.5} + \frac{1}{2.5} + \frac{1}{3.5} + \dots + \frac{1}{(N-0.5)}$	2.266511	2.959346	3.652416	4.345543

[3]

(iv) Find the differences  $\alpha_2 - \alpha_1, \alpha_3 - \alpha_2, \alpha_4 - \alpha_3$ . Show that the differences reduce by a factor of approximately 4 as  $N$  is doubled. Hence obtain the best estimate you can of  $k$ , giving your answer to an appropriate number of significant figures. [7]

**Mark Scheme 4776**  
**January 2006**

MEI Numerical Methods 4776

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**1** Use binomial expansion of  $(1 + r)^{-1}$  or sum of GP, or  $1 - r^2$  with  $r^2$  taken to be zero to obtain given result. [M1A1]  
 Relative error in reciprocal is of same magnitude but opposite sign [E1E1]  
 E.g. 10 is approx 2% greater than 9.8  
 $1/10 = 0.1$  is approx 2% less than  $1/9.8 = 0.10204$  [M1A1]  
[TOTAL 6]

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**2(i)**  $x_{r+1} = 1/\sin(x_r)$  [M1]  

	r	0	1	2	3	10	11	12	
	$x_r$	1	1.188395	1.077852	1.135147	1.113855	1.114323	1.114067	[M1A1]

 root is 1.11 to 3 sf. [A1]  
[subtotal 4]

**(ii)**

	x	2.7725	2.7735		
	$1/x - \sin(x)$	-8.4E-05	0.000719	change of sign, so 2.773 correct to 3 dp	

[M1A1A1]  
[subtotal 3]  
[TOTAL 7]

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**3**

	h	M	T		
	2	2.60242	2.44866		
	1	2.56982		$T2 = (M1 + T1)/2 =$	2.52554
				$T4 = (M2 + T2)/2 =$	2.54768
				$S1 = (2M1 + T1)/3$	
				=	2.55117
				$S2 = (2M2 + T2)/3$	
				=	2.55506

[M1A1]  
[A1]  
[M1A1]  
[A1]  
 I = 2.56 (or 2.555) is justified [A1]  
[TOTAL 7]

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**4**

	x	1	2	3	4	5
	$f(x)$	3	4.5	5.4	6.2	6.7

  

		<b>(i)</b>	<b>(ii)</b>		
	h	$f'(3)$	min	max	(i)
	2	0.925	0.9	0.95	(ii)
	1	0.85	0.8	0.9	[M1A1A1]

[M1A1A1]

Larger h gives smaller interval (or 0.9 is the only common value) [E1]  
 $f'(3) = 0.9$  is the value that seems justified. [E1]  
 (Or 0.8 seems to be the limit the process is tending to. [E1E1])  
[TOTAL 8]

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<b>5</b>	$x$	1	3	4		
	$g(x)$	4	1	11		
	$L(2) = \frac{4(2-3)(2-4)}{(1-3)(1-4)} + \frac{1(2-1)(2-4)}{(3-1)(3-4)} + \frac{11(2-1)(2-3)}{(4-1)(4-3)}$					[M1A1A1A1]
	$= -1^{1/3}$					[A1]
	$x$	$g(x)$	$\Delta g(x)$	$\Delta^2 g(x)$		
	1	4	-5.33333	7.666666		
		-				
	2	1.33333	2.33333	7.666667	2nd differences constant	
	3	1	10		so correct for a quadratic	[M1A1E1]
	4	11				
						<b>[TOTAL 8]</b>

<b>6 (i)</b>	$y' = 10x^9 - 10 = 0$ only when $x = 1$ , hence at most one turning point tenth degree polynomial is positive as $x$ tends to plus or minus infinity (hence exactly one turning point) (or other methods)					[M1A1] [E1]		
	$x$	0	1	2				
	$f(x)$	1	-8	1005		[M1A1]		
	changes of sign so roots in $[0,1]$ and $[1,2]$					[E1]		
	since only one turning point cannot be any more roots					[E1]		
						<i>[subtotal 7]</i>		
<b>(ii)</b>	NR:	$x_{r+1} = x_r - \frac{(x_r^{10} - 10x_r + 1)}{(10x_r^9 - 10)}$				[M1A1A1]		
	$r$	0	1	2	3	4	5	
	$x_r$	1.2	1.315589	1.284353	1.280004	1.279928	1.279928	[M1A1A1]
							<b>1.2799</b>	[subtotal 6]
<b>(iii)</b>	$x_{r+1} = (x_r^{10} + 1)/10$ If $x_0 = 0.1$ then $x_1 = (0.1^{10} + 1)/10 = 0.1^{11} + 0.1$ This is so close to 0.1 that further iterations are unnecessary					[M1A1] [M1A1] [E1]		
						<i>[subtotal 5]</i>		
							<b>[TOTAL 18]</b>	





## 4776: Numerical Methods

### General Comments

Many candidates seemed less well prepared for an examination in Numerical Methods than in recent sessions. There was evidence of whole topics being unfamiliar and of weaknesses in algebraic manipulation.

### Comments on Individual Questions

- 1) (Error analysis)  
This question proved to be beyond the grasp of the majority of candidates. Only a handful could produce any sort of algebraic argument to relate the relative errors in a number and its reciprocal. Very few were able to comment sensibly on what the given algebraic result represented. In many cases, only the arithmetic scored any marks.
- 2) (Solution of an equation)  
The iteration leading to the first root was carried out successfully in most cases (though sometimes the accuracy was not as required). In part (ii) a substantial minority did not appreciate the need to show a change of sign in the interval  $(2.7725, 2.7735)$ . Simply showing that  $f(2.773)$  is very near to zero scores no marks.
- 3) (Numerical integration)  
Most candidates were able to get one trapezium rule estimate and one Simpson's rule estimate correct. It was much less common to find two of each correct because of uncertainty about the relationships between the methods and a lack of clarity about the value of  $h$ . The degree of accuracy in the final answer was often inappropriate.
- 4) (Numerical differentiation)  
The point estimates and the ranges in which they lie were generally found accurately. The interpretation was a little more difficult, but there were several acceptable conclusions available for full credit. (This is one of those situations in numerical work in which there can be no definitive right answer to a question.)
- 5) (Lagrange's method and difference tables)  
Though substantial numbers of candidates were able to score full marks here, there were also many for whom this seemed to be an unknown topic. In Lagrange's formula it was a common error to confuse the  $x$  values and the function values. In the difference table it is necessary to say something about second differences being constant for a quadratic in order to gain full marks.

- 6) (Location of roots, solution of an equation)  
Part (i) of this question required a careful analysis of the nature of the function in order to identify the number of roots and their locations. A good many candidates showed that there were two roots in the interval  $(0, 2)$ , but failed to show convincingly that there are no other roots.  
In part (ii), the Newton-Raphson method was done well by most, though occasionally there was no evidence of any working whatever. Since some calculators have a solver facility, it is most unwise of a candidate to give no evidence of method.  
In part (iii) most could obtain  $0.1^{11} + 0.1$  as an iterate, but the argument that this value will be very close to the true root was given by almost none.
- 7) (Mid point rule, errors)  
In part (i) most candidates offered some explanation for the mid-point rule, but a minority merely re-asserted what was given in the question. The values of  $\log_e$  were sometimes wrong because the underlying laws of logarithms were not understood – though candidates often redeemed themselves by getting the values directly from their calculators in later work. Those who calmly and systematically followed instructions in parts (iii) and (iv) were generally successful.