

**ADVANCED SUBSIDIARY GCE
MATHEMATICS (MEI)**

4776/01

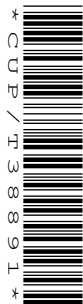
Numerical Methods

MONDAY 16 JUNE 2008

Afternoon

Time: 1 hour 30 minutes

Additional materials: Answer Booklet (8 pages)
Graph paper
MEI Examination Formulae and Tables (MF2)



INSTRUCTIONS TO CANDIDATES

- Write your name in capital letters, your Centre Number and Candidate Number in the spaces provided on the Answer Booklet.
- Read each question carefully and make sure you know what you have to do before starting your answer.
- Answer **all** the questions.
- You are permitted to use a graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.

INFORMATION FOR CANDIDATES

- The number of marks for each question is given in brackets [] at the end of each question or part question.
- The total number of marks for this paper is **72**.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.

This document consists of **4** printed pages.

Section A (36 marks)

- 1 The equation $f(x) = 0$ is known to have a single root in the interval $(3, 3.5)$. Given that $f(3) = 0.5$ and $f(3.5) = -0.8$, estimate the root using linear interpolation.

State the maximum possible error in this estimate. [6]

- 2 The function $f(x)$ has the values shown in the table. The value of k is to be determined.

| | | | | | |
|--------|---|---|---|-----|---|
| x | 1 | 3 | 5 | 7 | 9 |
| $f(x)$ | 2 | 1 | 5 | k | 2 |

Use a difference table to obtain the value of k , assuming that $f(x)$ is a cubic. [6]

- 3 The function $f(x) = \sqrt{1 + 3^x}$ is to be differentiated numerically.

Use the central difference method with $h = 0.2$ to estimate the derivative at $x = 2$. Obtain further estimates with $h = 0.1$ and $h = 0.05$.

By considering the differences between successive estimates, find the value of the derivative to an accuracy of 3 decimal places. [8]

- 4 Show that a Newton-Raphson iteration to find the cube root of 25 is

$$x_{r+1} = x_r - \frac{x_r^3 - 25}{3x_r^2}.$$

Perform three steps of this iteration, beginning with $x_0 = 4$. Show, by considering the differences between successive iterates, that the convergence is faster than first order. [8]

- 5 (i) Find $\sin 86^\circ - \sin 85^\circ$ to the accuracy given by your calculator. [1]

- (ii) A simple spreadsheet works to an accuracy of 6 significant figures. All intermediate answers used in calculations are rounded to 6 significant figures.

Write down the values of $\sin 86^\circ$ and $\sin 85^\circ$ as given by this spreadsheet. Hence find the value the spreadsheet gives for $\sin 86^\circ - \sin 85^\circ$. [3]

- (iii) You are now *given* that $\sin 86^\circ - \sin 85^\circ = 2 \cos 85.5^\circ \sin 0.5^\circ$. Find the value the spreadsheet gives for this expression. [2]

- (iv) Use your working from parts (ii) and (iii) to explain how two expressions that are mathematically identical can nevertheless evaluate differently. [2]

Section B (36 marks)

6 The integral $\int_1^3 \sqrt{1 + \sin x} \, dx$, where x is in radians, is to be evaluated numerically.

(i) Copy and complete the following table. [7]

| h | Mid-point rule estimate | Trapezium rule estimate |
|-----|-------------------------|-------------------------|
| 2 | $M_1 = 2.763\ 547$ | $T_1 =$ |
| 1 | $M_2 =$ | $T_2 =$ |
| 0.5 | $M_4 =$ | $T_4 =$ |

(ii) Show that the differences between successive mid-point rule estimates reduce by a factor of about 4.

State a result about the differences between successive trapezium rule estimates. [4]

(iii) Now let $S_1 = \frac{1}{3}(2M_1 + T_1)$, with S_2 and S_4 defined similarly.

Calculate S_1, S_2, S_4 and the differences $S_2 - S_1, S_4 - S_2$. By considering these differences, give the value of the integral to the accuracy that appears justified. [7]

7 The equation $x^2 = 4 + \frac{1}{x}$ has three roots.

(i) Show graphically that the equation has exactly one root for $x > 0$. Find the integer a such that this positive root lies in the interval $(a, a + 1)$.

Use the fixed-point iteration

$$x_{r+1} = \sqrt{\left(4 + \frac{1}{x_r}\right)} \quad (*)$$

to determine the positive root correct to 4 decimal places. [7]

(ii) The equation also has two negative roots. Without doing any calculations, explain why the iteration (*) cannot be used to find these negative roots.

Use the fixed-point iteration

$$x_{r+1} = -\sqrt{\left(4 + \frac{1}{x_r}\right)} \quad (**)$$

to find a negative root near to $x = -2$ correct to 4 decimal places. [5]

(iii) The third root of the equation lies in the interval $(-1, 0)$. Show that the iteration (**) used in part (ii) will not converge to this third root. Use another fixed point iteration to find the third root correct to 4 decimal places. [6]

Permission to reproduce items where third-party owned material protected by copyright is included has been sought and cleared where possible. Every reasonable effort has been made by the publisher (OCR) to trace copyright holders, but if any items requiring clearance have unwittingly been included, the publisher will be pleased to make amends at the earliest possible opportunity.

OCR is part of the Cambridge Assessment Group. Cambridge Assessment is the brand name of University of Cambridge Local Examinations Syndicate (UCLES), which is itself a department of the University of Cambridge.

4776 Numerical Methods

| | | | | | |
|---|------|-----|------|--|-----------|
| 1 | x | 3 | 3.5 | root = $(3 \times (-0.8) - 3.5 \times 0.5) / (-0.8 - 0.5)$ | [M1A1A1] |
| | f(x) | 0.5 | -0.8 | = .192308 (3.192, 3.19) | [A1] |
| | | | | (-) mpe is $3.5 - 3.192308 = 0.307602$ (0.308, 0.31) | [M1A1] |
| | | | | | [TOTAL 6] |

| | | | | | | |
|---|---|---|--------------|---------------|-------|------------|
| 2 | 1 | 2 | | | | |
| | 3 | 1 | -1 | | | |
| | 5 | 5 | 4 | 5 | | |
| | 7 | k | k-5 | k-9 | k-14 | |
| | 9 | 2 | 2-k | 7-2k | 16-3k | [M1A1A1A1] |
| | | | 16-3k = k-14 | hence k = 7.5 | | [M1A1] |
| | | | | | | [TOTAL 6] |

| | | | | | | |
|---|------|---------|---------|---------|----------|--|
| 3 | h | f(2+h) | f(2-h) | f'(2) | | derivatives |
| | 0.2 | .494507 | .867869 | .566594 | | [M1A1A1A1] |
| | 0.1 | .323418 | .010586 | .564163 | -0.00243 | differences |
| | 0.05 | .241636 | .085281 | .563555 | -0.00061 | [M1A1] |
| | | | | | | differences reducing by a factor 4 so next estimate about 1.56340. |
| | | | | | | 1.563 secure to 3 dp. |
| | | | | | | [M1] |
| | | | | | | [B1] |
| | | | | | | [TOTAL 8] |

| | | | | | | |
|---|---|----------------|---------|---------|----------|---|
| 4 | $f(x) = x^3 - 25$ | $f'(x) = 3x^2$ | | | | [M1A1A1] |
| | $x_{r+1} = x_r - (x_r^3 - 25) / 3x_r^2$ | (a.g.) | | | | |
| | r | 0 | 1 | 2 | 3 | |
| | x_r | 4 | 3.1875 | .945197 | 2.92417 | [M1A1] |
| | diffs | | -0.8125 | -0.2423 | -0.02103 | [B1] |
| | ratios | | | .298219 | .086783 | [B1] |
| | | | | | | [E1] |
| | | | | | | differences reducing at an increasing rate (<i>hence faster than first order</i>) |
| | | | | | | [TOTAL 8] |

| | | | | | |
|-------|---|-------------------------|--|--------|-----------|
| 5 (i) | 0.001 369 352 | (accept 0.001 369 4) | | [B1] | |
| (ii) | $\sin 86^\circ = 0.997$ | $\sin 85^\circ = 0.996$ | | [B1B1] | |
| | 564 | 195 | | | |
| | $\sin 86^\circ - \sin 85^\circ = 0.001 369$ | | | [A1] | |
| (iii) | $2 \times 0.0784591 \times 0.008 726 54$ | | | [M1] | |
| | = 0.00136935 | | | [A1] | |
| (iv) | Rounding has different effects in the two expressions (<i>may be implied</i>) | | | [E1] | |
| | First method involves subtraction of nearly equal numbers and so loses accuracy | | | [E1] | |
| | | | | | [TOTAL 8] |

| | | | | | |
|--------------|-----|-----------------|-----------------|------------|--|
| 6 (i) | h | M | T | | |
| | 2 | 2.763547 | 2.425240 | | |
| | 1 | 2.677635 | 2.594393 | mid-point: | [M1A1A1] |
| | 0.5 | 2.656743 | 2.636014 | trapezium: | [M1A1A1A1] [subtotal 7] |

| | | | | | |
|-------------|----|----------|--|--|-----------------|
| (ii) | M: | 2.763547 | diffs | | |
| | | 2.677635 | -0.08591 | | |
| | | 2.656743 | -0.02089 | reducing by a factor 4 (<i>may be implied</i>) | [M1A1E1] |
| | | | Differences in T reduce by a factor 4, too | [B1] [subtotal 4] | |

| | | | | | | |
|-------------------|--|----------|----------|--|------------------------------------|-------------------------|
| (iii) | | M | T | S | | |
| | | 2.763547 | 2.425240 | 2.650778 | | [M1] |
| | | 2.677635 | 2.594393 | 2.649888 | -0.00089033 | S values: [A1A1] |
| | | 2.656743 | 2.636014 | 2.649833 | -0.000054333 | diffs [A1] |
| | | | | Differences in S reducing fast e.g by a factor of (about) 16 | [E1] | |
| | | | | How this leads to an answer, e.g: | | |
| | | | | Next difference about -0.0000034 and/or next answer about 2.649830 | [E1] | |
| | | | | Accept 2.6498 or 2.64983 | [A1] [subtotal 7] | |
| [TOTAL 18] | | | | | | |

7 (i) Eg: graph of x^2 and $4 + 1/x$ for $x > 0$ showing single intersection **[G2]**
Change of sign to find interval (2,3) - i.e. $a = 2$ **[B1]**

| | | | | | | | |
|-------|-----|----------|----------|----------|---------|----------|------------------------------------|
| r | 0 | 1 | 2 | 3 | 4 | 5 | |
| x_r | 2.5 | 2.097618 | 2.115829 | 2.114859 | 2.11491 | 2.114907 | [M1A1A1] |
| | | | | | | | [A1] [subtotal 7] |
| | | | | | | | 2.1149 secure to 4 dp |

(ii) The iteration gives positive values only. **[E1]**

| | | | | | | | |
|-------|----|----------|----------|----------|----------|----------|------------------------------------|
| r | 0 | 1 | 2 | 3 | 4 | 5 | |
| x_r | -2 | -1.87083 | -1.86158 | -1.86087 | -1.86081 | -1.86081 | [M1A1A1] |
| | | | | | | | [A1] [subtotal 5] |
| | | | | | | | -1.8608 secure to 4 dp |

(iii) Eg

| | | | | | | |
|-------|------|----------|----------|----------|----------|--|
| r | 0 | 1 | 2 | 3 | 4 | |
| x_r | -0.5 | -1.41421 | -1.81463 | -1.85713 | -1.86052 | |

not converging to required root (converging to previous root) **[M1A1]**

Eg $x_{r+1} = 1 / (x_r^2 - 4)$ **[M1]**

| | | | | | | | |
|-------|------|----------|----------|----------|---------|---------|------------------------------------|
| r | 0 | 1 | 2 | 3 | 4 | 5 | |
| x_r | -0.5 | -0.26667 | -0.25452 | -0.25412 | -0.2541 | -0.2541 | [M1A1] |
| | | | | | | | [A1] [subtotal 6] |
| | | | | | | | -0.2541 secure to 4 dp |

[TOTAL 18]

4776 Numerical Methods (Written Examination)

General Comments

There were many good scripts seen, though as usual there were some candidates who appeared to be quite unprepared for this paper. The best candidates presented their work clearly and compactly, with due regard for the algorithmic nature of the subject. At the other extreme, some candidates presented their work as a jumble of figures, difficult to follow and frequently riddled with errors. It is worth saying yet again that candidates who adopt the latter approach put themselves at a considerable disadvantage.

Comments on individual questions

1) False position method

The single step of the false position method was surprisingly often done incorrectly. Even those who got the right answer were often unable to give the maximum possible error. Though this is elementary material the average mark scored was not high.

2) Difference table

This is elementary material too, but it defeated some. There were many errors of algebra with a common disregard of the rules for subtracting negative numbers – surely unforgivable at this level. Those who completed the difference table were sometimes unaware of the fact that, for a cubic, third differences will be constant. The more laborious methods used to find k were generally not successful.

3) Central difference method

The numerical values were generally found accurately, though some candidates worked with the forward difference formula. The differences and the ratio of differences then led in most cases to the right answer.

4) Newton-Raphson method

The Newton-Raphson method was familiar to most, though a fair number were unable to complete the simple derivation of the formula. The numbers were generally processed correctly. The explanation of the fact that the method is faster than first order was usually poor (or non-existent). The point is that the *ratio of differences* is decreasing rapidly. Far too many candidates said that the *differences* are decreasing rapidly.

5) Rounding errors

Once again, the numbers were generally handled well. Some candidates, however, worked in radians and thereby lost the point of the question. The final part was often not done well and only a handful of answers gained both marks. There are two points to be made here. Firstly, there is the general observation that using mathematically equivalent expressions may result in rounding being performed in different ways. Secondly, there is the specific observation that one of the methods used involves the subtraction of nearly equal quantities, something that is well known for losing significant figures.

6) Numerical integration

Most candidates scored well on this question. The values of M , T and S were usually correct. Showing that the differences in M values reduce by a factor of 4 as h is halved was well done by most. Some then demonstrated rather than stated that a similar result holds for T . This got full marks, of course, though it took up candidates' time.

Candidates were required to use the differences and ratio of differences in the S values to justify their final answers, and many (though not all) did so.

7) Finding the roots of an equation

Again, the numbers were handled well in this question, though finding the third root without an iterative formula being given did defeat some. The only part of the question that caused trouble was the comment at the start of part (ii). The iterative formula given is in the form of a square root. The correct comment was that a square root is, by definition, positive so it cannot give a negative answer. A common incorrect comment was that it is impossible to take the square root of a negative quantity.