RECOGNISING ACHIEVEMENT

ADVANCED SUBSIDIARY GCE

MATHEMATICS (MEI) 4776

Numerical Methods

Candidates answer on the Answer Booklet

OCR Supplied Materials:
- 8 page Answer Booklet
- Graph paper
- MEI Examination Formulae and Tables (MF2)

Other Materials Required:
None

Tuesday 13 January 2009
Morning

Duration: 1 hour 30 minutes

OCE/V02215

4776

INSTRUCTIONS TO CANDIDATES

- Write your name clearly in capital letters, your Centre Number and Candidate Number in the spaces provided on the Answer Booklet.
- Use black ink. Pencil may be used for graphs and diagrams only.
- Read each question carefully and make sure that you know what you have to do before starting your answer.
- Answer **all** the questions.
- Do **not** write in the bar codes.
- You are permitted to use a graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets **[ ]** at the end of each question or part question.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.
- The total number of marks for this paper is **72**.
- This document consists of **4** pages. Any blank pages are indicated.

Turn over

Section A (36 marks)

1 **(i)** Show by means of a difference table that a quadratic function fits the following data points.

x	–3	–1	1	3
y	–16	–2	4	2

[3]

(ii) Obtain the equation of the quadratic function, expressing your answer in its simplest form. **[5]**

2 **(i)** Use the formula for the difference of two squares to show that

$$\left(\sqrt{x+1}-\sqrt{x}\right)\left(\sqrt{x+1}+\sqrt{x}\right)=1. \qquad (*)$$

[2]

(ii) A spreadsheet shows $\sqrt{50001}$ as 223.6090 and $\sqrt{50000}$ as 223.6068.

Use the spreadsheet figures to obtain values of $\sqrt{50001}-\sqrt{50000}$

(*A*) by subtraction,

(*B*) by using (*)

Comment on your results. **[5]**

3 **(i)** For the integral

$$I=\int_0^{0.8}\sqrt{1-x^5}\,\mathrm{d}x$$

find the trapezium rule and mid-point rule estimates with $h = 0.8$ in each case. Use these estimates to obtain a Simpson's rule estimate. **[4]**

(ii) Given that the mid-point rule estimate with $h = 0.4$ is 0.784 069 to 6 significant figures, obtain a second Simpson's rule estimate. Without doing any further calculations, give a value for I to the accuracy that is justified. **[4]**

4 **(i)** An approximation to $\cos x$, where x is small and in radians, is given by

$$\cos x \approx 1 - 0.5x^2.$$

Find the absolute and relative errors in this approximation when $x = 0.3$. **[4]**

(ii) The formula

$$\cos x \approx 1 - 0.5x^2 + kx^4$$

gives a better approximation if k is suitably chosen. By considering $x = 0.3$ again, estimate k. **[2]**

5 A student is investigating the iteration

$$x_{r+1} = x_r^2 - 3x_r + 3$$

for different starting values x_0.

Determine the values of x_1 and x_2 in each of the cases $x_0 = 3$, $x_0 = 2.99$, $x_0 = 3.01$.

Evaluate the derivative of $x^2 - 3x + 3$ at $x = 3$.

Comment on your results. **[7]**

Section B (36 marks)

6 **(i)** Show that the equation

$$\sqrt{\sin x} + \sqrt{\cos x} = 1.5, \qquad (*)$$

where x is in radians, has a root in the interval (0.2, 0.3).

Perform two iterations of the bisection method and give the interval within which the root lies, the best estimate of the root, and the maximum possible error in that estimate. **[6]**

(ii) Now perform two iterations of the secant method, starting with $x_0 = 0.2$ and $x_1 = 0.3$. Give an estimate of the root to an appropriate number of significant figures.

Comment on the relative rate of convergence of the bisection method and the secant method. **[6]**

(iii) You are given that equation (*) also has a root α which is 1.298 504 to 6 decimal places. An iteration to find this root produces the following sequence of values.

r	0	1	2	3	4
x_r	1.4	1.314 351	1.298 887	1.298 504	1.298 504

By considering the values of $x_r - \alpha$, show that this iteration displays second order convergence making it clear what that means. **[6]**

[Question 7 is printed overleaf.]

7 A function f(x) has values, correct to 6 significant figures, as given in the table.

x	–0.4	–0.2	–0.1	0	0.1	0.2	0.4
f(x)	0.601 201	0.711 982	0.765 298	0.816 603	0.865 314	0.911 308	0.994 506

(i) Obtain three estimates of $f'(0)$ using the forward difference method with h equal to 0.4, 0.2, 0.1. Show that the differences between these estimates are approximately halved as h is halved. **[4]**

(ii) Obtain three estimates of $f'(0)$ using the central difference method. Show, by considering the differences between these estimates, that the central difference method converges more rapidly than the forward difference method. **[4]**

(iii) D_1 and D_2 are two estimates of a quantity d.

(A) Suppose that the error in D_2 is approximately half of the error in D_1. Write down expressions for the errors in D_1 and D_2 and hence show that $d \approx 2D_2 - D_1$.

(B) Now suppose that the error in D_2 is approximately a quarter of the error in D_1. Show that $d \approx \dfrac{4D_2 - D_1}{3}$. **[5]**

(iv) Use the results in part **(iii)**(A) and part **(iii)**(B) to obtain two further estimates of $f'(0)$. Give an estimate of $f'(0)$ to the accuracy that you consider justified. **[5]**

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1(i)

x	y	1st diff	2nd diff		
-3	-16				
-1	-2	14			
1	4	6	-8		**[M1A1]**
3	2	-2	-8	2nd difference constant so quadratic fits	**[E1]**

(ii) f(x) = -16 + 14(x + 3)/2 - 8(x + 3)(x + 1)/8 **[M1A1A1A1]**

= $-16 + 7x + 21 - x^2 - 4x - 3$

= $2 + 3x - x^2$ **[A1]**

[TOTAL 8]

2(i) Convincing algebra to demonstrate result **[M1A1]**

(ii)(A) Direct subtraction: 0.0022 **[B1]**

(B) Using (*): 1/(223.6090+223.6068) = 0.002236057 **[M1A1]**

Second value has many more significant figures ("more accurate") -- may be implied **[E1]**

Subtraction of nearly equal quantities loses precision **[E1]**

[TOTAL 7]

3(i)

x	f(x)			
0	1			
0.8	0.819951	T1 =	0.72798	**[M1]**
0.4	0.994867	M1 =	0.795893	**[M1]**
		hence S1 =	0.773256	**[M1]**

all values **[A1]**

(ii) T2 = 0.761937 **[B1]**

M2 = 0.784069 so S2 = 0.776692 **[M1A1]**

S2 will be much more accurate than S1 so 0.78 or 0.777 would be justified **[A1]**

[TOTAL 8]

4(i)

x	cosx	$1 - 0.5x^2$	error	rel error		
0.3	0.955336	0.955	-0.000336	-0.000352	*condone signs here but require correct sign for k*	**[M1A1A1A1]**

(ii) want k 0.3^4 = 0.000336 **[M1]**

gives k = 0.041542 (0.0415, 0.042, 1/24) **[A1]**

[TOTAL 6]

5

r	0	1	2
x_r	3	3	3
x_r	2.99	2.9701	2.911194
x_r	3.01	3.0301	3.091206

[M1A1A1]

Derivative is 2x - 3. Evaluates to 3 at x = 3 **[M1A1]**

3 is clearly a root, but the iteration does not converge **[E1]**

Need -1 < g'(x) < 1 at root for convergence **[E1]**

[TOTAL 7]

6(i) Demonstrate change of sign (f(a), f(b) below) and hence existence of root **[B1]**

a	b	f(a)	f(b)	x	mpe	f(x)	
0.2	0.3	-0.06429	0.021031	0.25	0.05	-0.01827	**[M1]**
0.25	0.3	-0.01827	0.021031	0.275	0.025	0.002134	**[M1]**
0.25	**0.275**			**0.2625**	**0.0125**	-0.00787	**[A1A1A1]**

[subtotal 6]

(ii)

r	x_r	f_r	
0	0.2	-0.06429	
1	0.3	0.021031	
2	0.275352	0.00241	**[M1A1]**
3	0.272161	-0.0001	**[M1A1]**

accept 0.27 or 0.272 as secure **[A1]**

secant method much faster **[E1]**

[subtotal 6]

(iii)

r	x_r	e_r	e_{r+1}/e_r^2		
0	1.4	0.101496		*e col:*	**[M1A1]**
1	1.314351	0.015847	1.538329	*e/e^2 col:*	**[M1A1]**
2	1.298887	0.000383	1.525122		
3	1.298504	= root			

equal values show 2nd order convergence **[E1]**

second order convergence: each error is proportional to the square of the previous error **[E1]**

[subtotal 6]

[TOTAL 18]

7(i) fwd diff:

h	0.4	0.2	0.1		
f '(0)	0.444758	0.473525	0.48711		**[M1A1A1]**
diffs		0.028768	0.013585	approx halved	**[B1]**

[subtotal 4]

(ii) cent diff:

h	0.4	0.2	0.1		
f '(0)	0.491631	0.498315	0.50008		**[M1A1A1]**
diffs		0.006684	0.001765	reduction greater than for forward difference	**[B1]**

[subtotal 4]

(iii) $(D_2 - d) = 0.5\ (D_1 - d)$ convincing algebra to $d = 2D_2 - D_1$ **[M1A1]**

$(D_2 - d) = 0.25\ (D_1 - d)$ convincing algebra to $d = (4D_2 - D_1)/3$ **[M1A1A1]**

[subtotal 5]

(iv) fwd diff: 2(0.48711) - 0.473525 = 0.500695 **[M1A1]**

cent diff: (4(0.50008) - 0.498315) / 3 = 0.500668 **[M1A1]**

0.5007 seems secure **[E1]**

[subtotal 5]

[TOTAL 18]

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General Comments

There was a lot of good work seen on this occasion, with few candidates appearing to be unready for the examination. Routine numerical calculations were generally carried out accurately, though it disappointing that so many candidates still set their work out so badly. An unsystematic layout is difficult to follow for the examiner and the candidate.

It was quite common to find that candidates could not interpret correctly what they were doing; in many cases, no attempt at all was made at parts of questions requiring interpretation.

Comments on individual questions

1) Difference table for a quadratic

The difference table was constructed correctly by the majority, though some were inconsistent in their signs. The quadratic is best found using Newton's formula, but some chose to use Lagrange. The majority of attempts were successful, but the algebra to simplify the result defeated quite a few.

2) Difference of square roots

The algebra in part (i) proved tricky to some, though many saw it immediately as the difference of squares formula. Some candidates failed to see that the required result could be obtained as the reciprocal of $\sqrt{50001} + \sqrt{50000}$. The comment expected in part (iii) was that a more accurate result can be obtained by avoiding subtracting nearly equal quantities. Despite the fact that this idea has been tested before it proved to be beyond many.

3) Numerical integration

The numerical values were generally found accurately and efficiently. Bizarrely, it was common for a final answer of 0.77669 to be rounded to 0.77.

4) Approximation to cos x

This question attracted many completely correct solutions. In particular, candidates seemed confident in their attempts at part (ii).

5) Fixed point iteration

This question was the most striking example of candidates carrying out the numerical calculations correctly but with little apparent understanding of what was going on. The iterations show that $x = 3$ is a root but that iterations starting near to $x = 3$ do not converge. The evaluation of the derivative shows that, at $x = 3$, the gradient of the function does not lie in the interval $[-1, 1]$. The link between these facts eluded the majority.

6) Numerical solution of an equation
The bisection method in part (i) was generally done well, though some candidates carried out too many or too few iterations. Others failed to state some of the required answers. The secant method in part (ii) was usually handled well also.

Part (iii), however, defeated many. They were asked to find the values of $x_r - \alpha$, but some chose to work with differences instead. Only a small minority seemed to know that, for second order convergence, $e_{r+1} \approx k\, e_r^2$.

7) Numerical differentiation
Once again, the numerical work was handled well in this question, with many candidates getting full marks in parts (i) and (ii). The algebra in part (iii) was seen to be very easy by some, but others spent a page more getting nowhere – or resorted to algebraic sleight of hand. The final part attracted quite a few good solutions, even from those who had not been able to do the algebra. However, it was quite common to see 0.500668 and 0.500695 rounded to 0.5006.