OCR
RECOGNISING ACHIEVEMENT

ADVANCED SUBSIDIARY GCE
MATHEMATICS (MEI) 4776/01
Numerical Methods

Candidates answer on the Answer Booklet

OCR Supplied Materials:
- 8 page Answer Booklet
- MEI Examination Formulae and Tables (MF2)
- Graph paper

Other Materials Required:
None

Wednesday 20 May 2009
Afternoon

Duration: 1 hour 30 minutes

INSTRUCTIONS TO CANDIDATES

- Write your name clearly in capital letters, your Centre Number and Candidate Number in the spaces provided on the Answer Booklet.
- Use black ink. Pencil may be used for graphs and diagrams only.
- Read each question carefully and make sure that you know what you have to do before starting your answer.
- Answer **all** the questions.
- Do **not** write in the bar codes.
- You are permitted to use a graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets **[ ]** at the end of each question or part question.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.
- The total number of marks for this paper is **72**.
- This document consists of **4** pages. Any blank pages are indicated.

Section A (36 marks)

1 A quadratic function, f(x), is to be fitted to the data shown in the table.

x	0	0.4	1
y	1.6	2.4	1.8

(i) Use Lagrange's method to find f(x), simplifying the coefficients. **[6]**

(ii) Explain why Newton's forward difference interpolation formula would not have been useful for this purpose. **[1]**

2 Show that the equation

$$x^2 + \frac{1}{x} = 3$$

has a root in the interval (1, 2).

Use the Newton-Raphson method to find this root, giving it correct to 6 significant figures. **[7]**

3 The numbers X and Y shown below are known to be correct to 3 decimal places.

$X = 2.718 \qquad Y = 3.142$

(i) State the maximum possible errors in X, $X + Y$, $X - Y$, $10X + 20Y$. **[4]**

(ii) Find the maximum possible relative errors in X and Y. Hence state approximately the maximum possible relative errors in XY and $\frac{X}{Y}$. **[4]**

4 You are given that, for A and B in radians and $A \approx B$,

$$\frac{\sin A - \sin B}{A - B} \approx \cos\frac{A + B}{2}. \qquad (*)$$

A computer program calculates values of sine and cosine correct to 6 decimal places.

(i) In the case $A = 1.01$, $B = 1$, find the values of the left and right sides of (*) as calculated by this program. **[2]**

(ii) Identify two distinct reasons for the difference in these two values. **[2]**

(iii) Explain briefly why the right side of (*) is likely to be evaluated more accurately than the left as A gets progressively closer to B. **[2]**

5 Sketch, on the same axes, the graphs $y = x$ and $y = 1 - x^4$ for $0 \leqslant x \leqslant 1$. You should use the same scale on each axis.

Show numerically that the iteration $x_{r+1} = 1 - x_r^{\ 4}$, starting with $x_0 = 0.6$, diverges.

Illustrate this divergence on your sketch, showing x_0, x_1, x_2, x_3. **[8]**

Section B (36 marks)

6 The integral $\int_0^{0.8} \sqrt{3+x-x^2}\,dx$ is to be evaluated numerically.

(i) Find, as efficiently as possible, the mid-point rule estimates and the trapezium rule estimates for $h = 0.8$ and 0.4. **[6]**

(ii) Use the values in part **(i)** to show that the first Simpson's rule estimate is 1.427 959 (correct to 6 decimal places), and to find a second Simpson's rule estimate. **[3]**

(iii) Given that, for $h = 0.2$, the mid-point rule estimate is 1.428 782 and the trapezium rule estimate is 1.426 497, calculate a third Simpson's rule estimate. **[2]**

(iv) Show that the differences between successive mid-point rule estimates reduce by a factor of about 0.25 as h is halved. Find the corresponding factor for the Simpson's rule estimates. Hence give the integral to the accuracy that appears justified. **[7]**

7 (i) Use Newton's forward difference interpolation formula to find the quadratic function that passes through the following data points.

x	1	1.2	1.4
f(x)	0.6	–0.1	0.4

[8]

(ii) Use the quadratic function to estimate $f'(1.2)$. Show that the central difference formula gives exactly the same estimate. What does this suggest about the central difference formula? **[5]**

(iii) Use the quadratic function to estimate $f'(1)$. Show that the forward difference does not give the same value. What does this show about the forward difference method? Which of these two estimates is likely to be more accurate? **[5]**

4776 Numerical Methods

1(i) f(x) = 1.6(x - 0.4)(x - 1)/(-0.4)(-1) + 2.4x(x - 1)/0.4(0.4 - 1) + 1.8x(x - 0.4)/1(1 - 0.4) **[M1A1,1,1]**

= 4(x^2 - 1.4x + 0.4) - 10(x^2 - x) + 3(x^2 - 0.4x) **[A1]**

= - 3x^2 + 3.2x + 1.6 **[A1]**

(ii) Newton's formula requires equally spaced data **[E1]**

[TOTAL 7]

2

x	1	2		
x^2 + 1/x - 3	-1	1.5	(change of sign so root)	**[M1A1]**

f(x) = x^2 + 1/x - 3 so f '(x) = 2x - 1/x^2 hence NR formula **[M1A1]**

r	0	1	2	3		
x_r	1.5	1.532609	1.532089	1.532089	**1.53209**	**[M1A1A1]**

[TOTAL 7]

3(i)

term	X	X+Y	X - Y	10X + 20Y	
mpe	0.0005	0.001	0.001	0.015	**[B1B1B1B1]**

(ii)

term	X	Y	XY	X/Y	
mpre	0.000184	0.000159	0.000343	0.000343	**[B1B1B1B1]**

[TOTAL 8]

4(i)

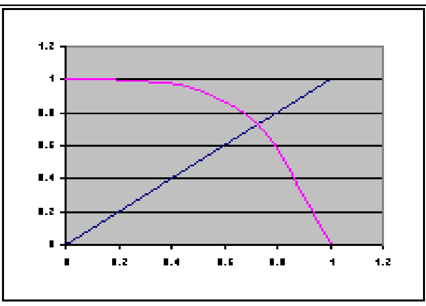
to 6 dp:	sin A	sin B	LHS	RHS	
	0.846832	0.841471	0.5361	0.536088	**[B1B1]**

(ii) It is an approximate equality. LHS involves subtraction of nearly equal numbers.
LHS involves 2 trig functions, RHS just 1. **[E1E1]**

(iii) Subtraction of nearly equal quantities is a bigger problem as the difference decreases.
RHS involves no such problem. **[E1E1]**

[TOTAL 6]

5



r	x_r
0	0.6
1	0.8704
2	0.426048
3	0.967052

[G2]

[M1A1A1]

cobweb diagram showing spiralling out from root **[M1A1A1]**

[TOTAL 8]

6(i)	x	f(x)						
	0	1.732051						
	0.8	1.777639			T1 =	1.403876	*M*	**[M1]**
	0.4	1.8	M1 =	1.44	T2 =	1.421938	*T values*	**[M1]**
	0.2	1.777639						**[A1,1,1,1]**
	0.6	1.8	M2 =	1.431056				
								[subtotal 6]
(ii)	S1 =	1.427959	(a.g.)					**[M1]**
	S2 =	1.428016						**[M1A1]**
								[subtotal 3]
(iii)	S4 =	(2 M4 + T4) / 3 =		1.428020				**[M1A1]**
								[subtotal 2]
(iv)	M	1.44	1.431056	1.428782				
	diffs		-0.00894	-0.00227				
	ratio			0.254186	approx 0.25			**[M1A1]**
	S	1.427959	1.428016	1.428020				
	diffs		5.77E-05	3.99E-06				
	ratio			0.069037	(approx 0.0625)			**[M1A1A1]**
	Reasoning to: integral is secure as 1.42802(0)							**[M1B1]**
								[subtotal 7]
								[TOTAL 18]

7(i)	x	f(x)	1st diff	2nd diff	
	1	0.6			
	1.2	-0.1	-0.7		
	1.4	0.4	0.5	1.2	**[M1A1]**
	$f(x) = 0.6 + (-0.7)(x - 1) / 0.2 + 1.2(x - 1)(x - 1.2) /(2 (0.2)^2)$				**[M1A1A1A1]**
	$= 0.6 - 3.5x + 3.5 + 15x^2 - 33x + 18$				
	$= 15x^2 - 36.5x + 22.1$				**[M1A1]**
					[subtotal 8]
(ii)	f '(x) = 30x - 36.5		f '(1.2) = 36 - 36.5 = -0.5		**[M1A1]**
	Central difference:		(0.4 - 0.6)/(1.4 - 1) = -0.2/0.4 = -0.5		**[M1A1]**
	Suggests central difference is accurate for quadratics.				**[E1]**
					[subtotal 5]
(iii)			f '(1) = 30 - 36.5 = -6.5		**[B1]**

Forward difference:	(-0.1 - 0.6)/(1.2 - 1) = -0.7/0.2 = -3.5	**[M1A1]**
Shows that forward difference is not exact for quadratics.		**[E1]**
Quadratic estimate (-6.5) is likely to be more accurate. (Allow comments saying that we cannot be sure.)		**[E1]** ***[subtotal 5]***
		[TOTAL 18]

1(i)

-1 < g'(α) < 1 **[B1]**

E.g. Multiply both sides of x = g(x) by λ and add (1 - λ)x to both sides. **[M1A1]**

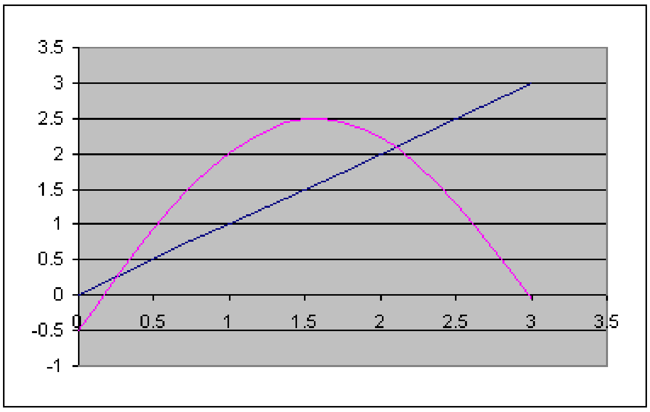
Derivative of rhs set to zero at root: λg'(α) + 1 - λ = 0 **[M1A1]**

algebra to obtain given result **[A1]**

In practice use an initial estimate x_0 in place of α **[A1]**

[subtotal 7]

(iii)



[G3]

Roots approximately 0.25, 2.1 **[B1B1]**

Eg:

r	x_r	x_r	x_r	x_r	x_r	x_r
0	0	0.2	0.4	2	2.2	2.4
1	-0.5	0.096008	0.668255	2.227892	1.925489	1.52639
2	-1.93828	-0.21242	1.358852	1.875308	2.31326	2.497043
3	-3.29971	-1.13247	2.432871	2.36198	1.710416	1.302517
4	-0.02763	-3.21639	1.452591	1.609012	2.470807	2.392685
5	-0.58289	-0.2758	2.479066	2.49781	1.364805	1.542517
6	-2.15131	-1.31696	1.345334	1.300676	2.436576	2.498801
7	-3.00855	-3.40387	2.424072	2.391217	1.444139	1.298298
8	-0.89795	0.277847	1.472555	1.545741	2.475969	2.389305
9	-2.84615	0.322857	2.485535	2.499058	1.352649	1.549934
10	-1.37349	0.451832	1.329994	1.297679	2.4289	2.499347

No convergence in each case **[M1A1A1]**

Let g(x) = 3 sinx - 0.5
Then g'(x) = 3 cosx
So λ = 1 / (1 - 3 cosα) **[M1A1]**

	Smaller root: λ =	-0.52446 (approx -0.5)		Larger root: λ =	0.397687 (approx 0.4)		**[M1A1A1]**
	r	x_r	NB: must be using relaxation	r	x_r		
	0	0.25		0	2.1		
	1	0.253894		1	2.09585 1		
	2	0.254078		2	2.09586 6		
	3	0.254087		3	2.09586 6		
	4	0.254088		4	2.09586 6		**[M1A1]**
	5	0.254088		5	2.09586 6		**[M1A1]**
							[subtotal 17] **[TOTAL 24]**

2(i)	f(x) = 1	2h = 2a + b	**[M1A1]**
	f(x) = x, x^3 give 0 = 0		**[M1A1]**
	f(x) = x^2	$2h^3/3 = 2a\alpha^2$	**[A1]**
	f(x) = x^4	$2h^5/5 = 2a\alpha^4$	**[A1]**
	Convincing algebra to verify given results		**[A1A1]** **[subtotal 8]**

		L	R	m	h	x1	x2		
(ii)		0	0.785398	0.392 699	0.3926 99	0.08851 6	0.69688 2		
	function values			1.189 207		1.04343 1	1.35535		**setup:**
	weights			0.349 066		0.21816 6	0.21816 6		**[M3A3]**
	integral			0.415 112		0.22764 1	0.29569 1	**0.9384 44**	**[A1]**
		L	R	m	h	x1	x2		
		0	0.392699	0.196 35	0.1963 5	0.04425 8	0.34844 1		
	function values			1.094 949		1.02190 3	1.16758 9		
	weights			0.174 533		0.10908 3	0.10908 3		
	integral			0.191 105		0.11147 2	0.12736 4	0.4299 41	
		0.392699	0.785398	0.589 049	0.1963 5	0.43695 7	0.74114		
	function values			1.291 58		1.21122 6	1.38390 1		**repeat:**
	weig			0.174		0.10908	0.10908		**[M2]**

hts			533		3	3		
inte			0.225		0.13212		0.5085	
gral			423		4	0.15096	08	
							0.9384	
							49	**[A1]**

Either repeat with h halved to verify that 0.938449 is correct to 6 dp **[M1A1]**
Or observe that the method is converging so rapidly that 0.938449 will be correct to 6dp **or [E1A1]**

[subtotal 12]

(iii) Use routine known to deliver 6dp and vary k:

							1.4657	
						k =	2	
	L	R	m	h	x1	x2		
			0.196	0.1963	0.04425	0.34844		
	0	0.392699	35	5	8	1		
			1.136		1.03194	1.23791		
function values			464		6	8		
weig			0.174		0.10908	0.10908		
hts			533		3	3		
inte			0.198		0.11256	0.13503	0.4459	
gral			35		8	6	54	
			0.589	0.1963	0.43695			
	0.392699	0.785398	049	5	7	0.74114		**modify**
			1.406		1.29791	1.53016		
function values			898		8	4		**[M1A1]**
weig			0.174		0.10908	0.10908		
hts			533		3	3		
inte			0.245		0.14158	0.16691	0.5540	
gral			55		1	5	46	
							1.0000	
							00	
	k	1.465	1.466	1.467				**find k**
			1.000	1.0001				
	integral	0.999908	036	63				**[M1A1]**

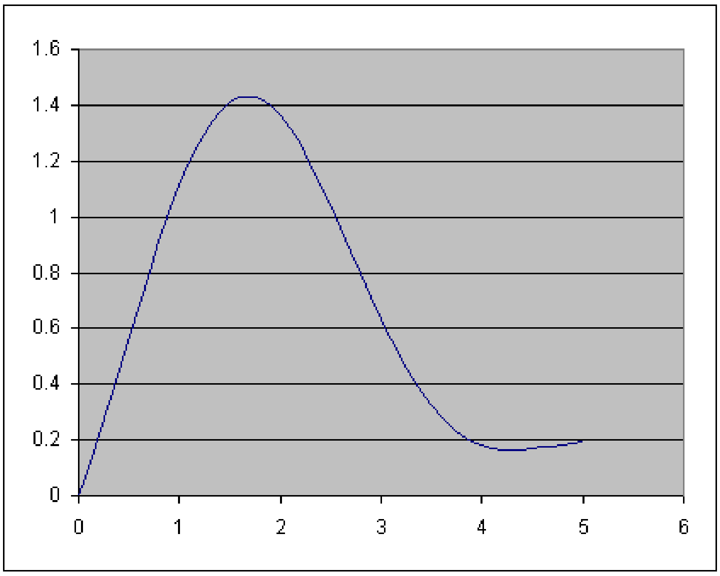
hence k = 1.466

[subtotal 4]
[TOTAL 24]

3(i)	Use central difference formulae for 2nd and 1st derivatives to obtain first given result	**[M1A1A1]**
	Hence obtain $y_1 = h^2 - y_{-1}$	**[M1A1]**
	Use central difference to obtain $y_1 - y_{-1} = 2h$	**[M1A1]**
	Hence given result for y_1	**[M1]** **[subtotal 8]**

(ii)

h	x	y
0.1	0	0
	0.1	0.105
	0.2	0.216472
	0.3	0.332426
	0.4	0.450961
	0.5	0.570174
	0.6	0.68815
	0.7	0.802981
	0.8	0.912793
	0.9	1.015786
	1	1.11027
	1.1	1.194705
	1.2	1.26774
	1.3	1.328248
	1.4	1.375354
	1.5	1.40846
	1.6	1.42726
	1.7	1.431751
	1.8	1.42223
	1.9	1.399287
	2	1.363785
	2.1	1.316838
	2.2	1.259773
	2.3	1.194096
	2.4	1.121445
	2.5	1.04354
	2.6	0.962141
	2.7	0.878993
	2.8	0.79578
	2.9	0.714082
	3	0.635337
	3.1	0.560807
	3.2	0.491549
	3.3	0.428404
	3.4	0.371982
	3.5	0.322662
	3.6	0.280597
	3.7	0.245729
	3.8	0.217808
	3.9	0.196416
	4	0.180999
	4.1	0.170894
	4.2	0.165365
	4.3	0.163635
	4.4	0.164915
	4.5	0.168435
	4.6	0.173469
	4.7	0.179352
	4.8	0.185502



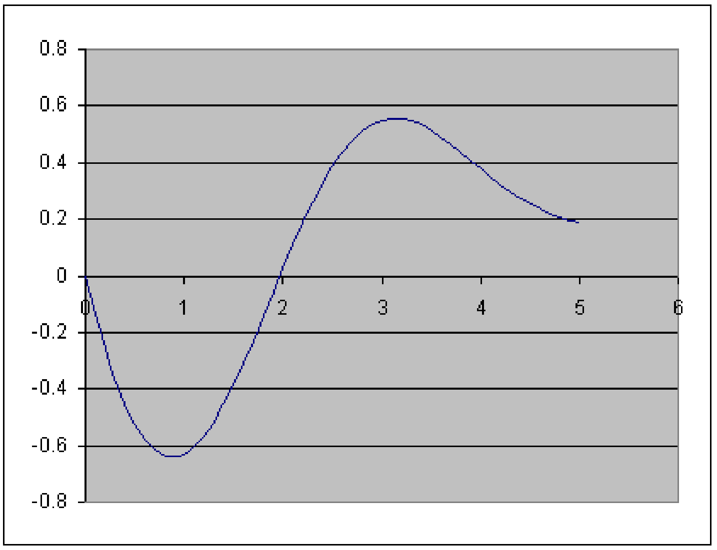
	4.9	0.191424
	5	0.196725

setup [M3] numbers [A3] graph [A3] [subtotal 9]

(ii) Obtain formula $y_1 = ah + 0.5h^2$ **[M1A1]**
Modify routine **[M1A1]**

Trial on a to obtain a = -1.4 or -1.5 **[M1A1G1]**

h	x	y
0.1	0	0
a	0.1	-0.135
-1.4	0.2	-0.25582
	0.3	-0.36107
	0.4	-0.44993
	0.5	-0.5219
	0.6	-0.57677
	0.7	-0.6146
	0.8	-0.63565
	0.9	-0.64047
	1	-0.6298
	1.1	-0.60462
	1.2	-0.56614
	1.3	-0.51572
	1.4	-0.45494
	1.5	-0.3855
	1.6	-0.3092
	1.7	-0.22792
	1.8	-0.14356
	1.9	-0.05802
	2	0.026884
	2.1	0.109408
	2.2	0.187962
	2.3	0.26113
	2.4	0.327696
	2.5	0.386672
	2.6	0.437316
	2.7	0.479135
	2.8	0.51189
	2.9	0.535589
	3	0.550471
	3.1	0.556986
	3.2	0.555768
	3.3	0.547604
	3.4	0.533401
	3.5	0.514147
	3.6	0.490876
	3.7	0.464631
	3.8	0.43643
	3.9	0.40724
	4	0.377942
	4.1	0.349319
	4.2	0.322033
	4.3	0.296623
	4.4	0.27349



4.5	0.252909	
4.6	0.235026	
4.7	0.219875	
4.8	0.207386	
4.9	0.197404	
5	0.189706	
		[subtotal 7]
		[TOTAL24]

4(i) Diagonal dominance: the magnitude of the diagonal element in any row is greater than or equal to the sum of the magnitudes of the other element. **[E1]**

| a | > | b | + 2 will ensure convergence. (> required as dominance has to be strict) **[E1E1]**

[subtotal 3]

(ii)

					a	b	
4	1	2	1	1	4	2	
1	4	1	2	0			
2	1	4	1	0			
1	2	1	4	0			
0	0	0	0				
0.25	-0.0625	-0.10938	-0.00391				**setup**
0.321289	-0.05103	-0.14691	-0.01808				**[M3A3]**
0.340733	-0.03941	-0.15599	-0.02648				
0.344469	-0.03388	-0.15715	-0.02989				
0.344515	-0.0319	-0.15681	-0.03098				
0.344124	-0.03134	-0.15648	-0.03124				
0.343886	-0.03123	-0.15633	-0.03127				
0.343789	-0.03123	-0.15627	-0.03127				
0.343758	-0.03124	-0.15625	-0.03126				
0.34375	-0.03125	-0.15625	-0.03125				
0.343749	-0.03125	-0.15625	-0.03125				
0.34375	-0.03125	-0.15625	-0.03125				**values**
0.34375	-0.03125	-0.15625	-0.03125				**[A3]**

					a	b	
2	1	4	1	1	2	4	
1	2	1	4	0			
4	1	2	1	0			
1	4	1	2	0			
0	0	0	0				
0.5	-0.25	-0.875	0.6875				
2.03125	-1.95313	-3.42969	4.605469				
6.033203	-10.5127	-9.11279	22.56519				
12.69934	-46.9236	-13.2195	94.10735				
3.347054	-183.278	37.89147	345.9377				
-156.613	-632.515	456.5137	1115.079				**values**
-1153.81	-1881.51	2690.835	2994.509				**[A3]**
-5937.67	-4365.6	12560.88	5419.593				

[subtotal 12]

(iii)	No convergence when a = 2, b = 0 Indicates that non-strict diagonal dominance is not sufficient	**[M1A1]** **[E1E1]** **[subtotal 4]**
(iv)	Use RHSs 1,0,0,0 0,1,0,0 0,0,1,0 0,0,0,1 to obtain inverse as 0.34375 -0.03125 -0.15625 -0.03125 -0.03125 0.34375 -0.03125 -0.15625 -0.15625 -0.03125 0.34375 -0.03125 -0.03125 -0.15625 -0.03125 0.34375	**[M1]** **[A1]** **[A1]** **[A1]** **[A1]** **[subtotal 5]** **[TOTAL 24]**

4776 Numerical Methods (Written Examination)

General Comments

There were many good scripts seen, and some were excellent. However, as usual, there were some candidates who appeared to be quite unprepared for this paper. The best candidates presented their work clearly and compactly, with due regard for the algorithmic nature of the subject. At the other extreme, some candidates presented their work as a jumble of figures, difficult to follow and frequently riddled with errors. It is worth saying yet again that candidates who adopt the latter approach put themselves at a considerable disadvantage.

Comments on Individual Questions

1) **Lagrange interpolation**

Most candidates know what to do, though some confused the x and f(x) values. The algebra to simplify the polynomial defeated far too many. In part (ii) most knew that Newton's formula requires equally spaced x values.

2) **Newton-Raphson method**

Almost everyone established the existence of the root correctly by using change of sign. The Newton-Raphson method is well understood and most could set up the iteration correctly and find the root to the required accuracy. A small number presented an answer without any working. As the rubric to the paper makes clear, this cannot be rewarded.

3) **Absolute and relative errors**

Though this is very elementary material, most candidates scored half marks or less and it was extremely rare for anyone to get full marks. The maximum possible errors are 0.0005, 0.001, 0.001, 0.015. It was quite common to see these values doubled. The maximum possible relative errors in X and Y were often found correctly. Some knew that the maximum possible relative error in XY will be the sum of the maximum possible relative errors in X and Y. Almost nobody knew that the maximum possible relative error in X/Y will *also* be the sum of the maximum possible relative errors in X and Y.

4) **Errors in evaluating a formula**

It was surprising that quite a few candidates failed to follow relatively simple instructions to work to 6 decimal places. Inevitably, some worked in degrees. The attempts at parts (ii) and (iii) were often poor. In part (ii) candidates might have said that the equality is only approximate, that the left side involves subtraction of nearly equal quantities, or that the left side involves two trigonometric evaluations while the right involves one. In part (iii) the point is that the problem of subtracting nearly equal quantities gets steadily worse on the left but there is no such problem on the right. This is a well worn topic on this paper, but many seemed not to know it.

5) **Fixed point iteration**

This question was frequently answered well. The graphs were not all objects of beauty, but they were used successfully by many to illustrate the divergence by means of a cobweb diagram. The error of going 'up to the line and across to the curve' was seen from time to time.

6) Numerical integration

Most candidates scored well on this question. The values of *M*, *T* and *S* were usually correct. Showing that the differences in *M* values reduce by a factor of 4 as *h* is halved was well done. Most could then show that the corresponding factor for *S* is about 16. Some candidates, despite having analysed the convergence of the *S* values, stated the final answer without any attempt to justify the number of figures given.

7) Newton's forward difference formula

Newton's formula was handled well by many, though the fact that $h = 0.2$ led to errors. In part (ii) the two estimates were often found correctly but the comments on the fact that they are equal were sometimes rather feeble: 'the central difference formula is quite accurate'. The best answer, building on what candidates should have learned, is that the central difference formula is exact for quadratics. In part (iii) the best conclusion is that the forward difference formula is not exact for quadratics.