

ADVANCED SUBSIDIARY GCE
MATHEMATICS (MEI)
Numerical Methods

4776/01

Candidates answer on the Answer Booklet

OCR Supplied Materials:

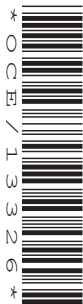
- 8 page Answer Booklet
- Graph paper
- MEI Examination Formulae and Tables (MF2)

Other Materials Required:

None

Friday 15 January 2010
Afternoon

Duration: 1 hour 30 minutes



INSTRUCTIONS TO CANDIDATES

- Write your name clearly in capital letters, your Centre Number and Candidate Number in the spaces provided on the Answer Booklet.
- Use black ink. Pencil may be used for graphs and diagrams only.
- Read each question carefully and make sure that you know what you have to do before starting your answer.
- Answer **all** the questions.
- Do **not** write in the bar codes.
- You are permitted to use a graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets [] at the end of each question or part question.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.
- The total number of marks for this paper is **72**.
- This document consists of **4** pages. Any blank pages are indicated.

Section A (36 marks)

- 1 Show that the equation

$$2^x + \left(\frac{1}{2}\right)^x = 3$$

has a root between $x = 1.3$ and $x = 1.5$. Use the bisection method to find an estimate of this root with a maximum possible error less than 0.02.

Determine how many further iterations would be required to reduce the maximum possible error to less than 0.001. [8]

- 2 An integral, $\int_a^b f(x) dx$, is being evaluated numerically. Some mid-point rule and trapezium rule estimates are shown in the table.

h	Mid-point rule	Trapezium rule
1	2.579 768	2.447 490
0.5	2.547 350	

Find the trapezium rule estimate for $h = 0.5$.

Find two Simpson's rule estimates and hence state, with a reason, the value of the integral to the accuracy that appears justified. [7]

- 3 (i) Given that $f(x) = x^3 - x^2 + 1$, find $f(0.5)$.

Use the formula $f(x + h) \approx f(x) + h f'(x)$ to show that

$$f(0.5 + h) \approx 0.875 - 0.25h. \quad [3]$$

- (ii) Hence determine the approximate range of values of x for which $f(x) = 0.875$ correct to 3 decimal places. [4]

- 4 (i) Show algebraically that

$$(k + 1)^2 + (k - 1)^2 - 2k^2 = 2 \quad (*)$$

for all values of k . [2]

- (ii) Use your calculator to evaluate the left hand side of (*) for increasingly large values of k (e.g. 10^3 , 10^6 , 10^9 , ...). State briefly two important results in numerical methods that are illustrated by your working. [4]

- 5 A function $f(x)$ has the following values correct to 3 decimal places.

x	0	1	2	3	4
$f(x)$	1.883	2.342	2.874	3.491	4.206

- (i) Show, by means of a difference table, that a cubic polynomial fits these data points closely but not exactly. [4]
- (ii) Use Newton's forward difference formula to estimate the value of $f(1.5)$. [4]

Section B (36 marks)

- 6 (i) The derivative of a function is to be estimated numerically. Show, with the aid of a sketch, that the central difference method will generally be more accurate than the forward difference method. [4]
- (ii) The table shows two values of $\tan x^\circ$ correct to 7 significant figures.

x	60	62
$\tan x^\circ$	1.732 051	1.880 726

Use these two values to estimate the derivative of $\tan x^\circ$ at $x = 60$.

Use your calculator to find two further estimates of this derivative, using the forward difference method and taking $h = 1$ and $h = 0.5$. [4]

- (iii) Use the central difference method with $h = 2$, $h = 1$ and $h = 0.5$ to obtain three estimates of the derivative of $\tan x^\circ$ at $x = 60$. [4]
- (iv) Show that the differences between the estimates in part (iii) reduce by a factor of about 0.5 as h is halved.

By considering the differences between the estimates in part (iii) show that the central difference method seems to converge more rapidly than the forward difference method. [6]

[Question 7 is printed overleaf.]

- 7 (i) Show, by means of a sketch or otherwise, that the equation

$$x = 3 \sin x, \quad (*)$$

where x is in radians, has a root, α , in the interval $(\frac{1}{2}\pi, \pi)$. Determine how many other non-zero roots, if any, the equation has. [3]

- (ii) Determine whether or not the iteration

$$x_{r+1} = 3 \sin x_r,$$

starting with $x_0 = 2$, converges to α . Illustrate your answer with a staircase or cobweb diagram as appropriate. [7]

- (iii) Show that equation (*) may be rearranged into the form

$$x = \sin x + \frac{2}{3}x.$$

Show that the corresponding iteration, starting with $x_0 = 2$, converges rapidly. State to 5 decimal places the value to which the iteration converges. Verify that this value for α is correct to 5 decimal places. [8]

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4776 Numerical Methods

1	x	LHS						
	1.3	2.868415	< 3					
	1.5	3.181981	> 3					[M1A1]
				mpe	<i>(may be implied)</i>			
	1.4	3.017945		0.1				[M1]
	1.35	2.941413		0.05				[A1]
	1.375	2.979232		0.025				[A1]
	1.3875			0.0125			<i>finishing at this point:</i>	[A1]
	mpe:	0.00625	0.003125	0.001563	0.000781	< 0.001	so 4 more iterations	[M1A1]
								[TOTAL 8]

2	h	M	T	S				
	1	2.579768	2.447490	2.535675	T		[M1A1]	
	0.5	2.547350	2.513629	2.536110	S		[M1A1A1]	
		2.536 secure by comparison of S values.						[E1A1]
							[TOTAL 7]	

3(i)	$f'(x) = 3x^2 - 2x$	so $f'(0.5) = -0.25$		[B1B1]
	$f(0.5) = 0.875$	hence given result		[B1]

(ii)	Require $-0.0005 < 0.25h < 0.0005$	[M1A1]
	Hence $-0.002 < h < 0.002$	[A1]
	And so $0.498 < x < 0.502$	[B1]
		[TOTAL 7]

4(i)	Convincing algebra to given result	[M1A1]
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(ii)	Eg $k = 1000$ correct evaluation to 2	[B1]
	$k = 1000000$ incorrect evaluation to zero (NB some will need larger k)	[B1]
	Mathematically equivalent expressions do not always evaluate equally	[E1]
	(because calculators do not store (large) numbers exactly)	
	Subtraction of nearly equal quantities often causes problems	[E1]
		[TOTAL 6]

5(i)	x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$		
	0	1.883					
	1	2.342	0.459			1st diff:	[M1A1]
	2	2.874	0.532	0.073		2nd, 3rd	[F1]
	3	3.491	0.617	0.085	0.012		
	4	4.206	0.715	0.098	0.013		
			3rd diffs almost constant				[E1]

(ii)	$f(1.5) = 1.883 + 0.459 \times 1.5 + 0.073 \times 1.5 \times 0.5 / 2! + 0.012 \times 1.5 \times 0.5 \times (-0.5) / 3!$	[M1A1A1]
	$f(1.5) = 2.598125$ or 2.598 to 3 dp	[A1]
		[TOTAL 8]

6 (i)	Sketch of smooth curve and its tangent.	[G1]
	Forward and central difference chords.	[G1G1]
	Clear statement or implication that the central difference chord has gradient closer to that of the tangent	[E1]
		[subtotal 4]

(ii)	h	$\tan 60^\circ$	$\tan (60 + h)^\circ$	derivative	
	2	1.732051	1.880726	0.074338	[M1A1]
	1	1.732051	1.804048	0.071997	[A1]
	0.5	1.732051	1.767494	0.070886	[A1]
					[subtotal 4]

(iii)	h	$\tan (60 + h)^\circ$	$\tan (60 - h)^\circ$	derivative	
	2	1.880726	1.600335	0.070098	[M1A1]
	1	1.804048	1.664279	0.069884	[A1]
	0.5	1.767494	1.697663	0.069831	[A1]
					[subtotal 4]

(iv)	forward difference:	derivative	diffs	ratio of diffs	
		0.074338			
		0.071997	-0.00234		
		0.070886	-0.00111	0.474407 (about 0.5, may be implied)	[M1A1A1]
	central difference:	derivative	diffs	ratio of diffs	
		0.070098			
		0.069884	-0.00021		
		0.069831	-5.3E-05	0.24896 (about 0.25, less than forward difference, hence faster)	[M1A1E1]
					[subtotal 6]
					[TOTAL 18]

- 7 (i) Sketch showing $y = 3 \sin x$ and $y = x$ with intersection in $(\frac{1}{2}\pi, \pi)$
State or show that there is only one other non-zero root

[G1G1]
[E1]
[subtotal 3]

- (ii)
- | | | | | | | |
|-------|---|----------|----------|---------|----------|---------|
| r | 0 | 1 | 2 | 3 | 4 | 5 |
| x_r | 2 | 2.727892 | 1.206001 | 2.80259 | 0.997639 | 2.52058 |
- clearly not converging
Cobweb diagram to illustrate process

[M1A1A1]
[B1]
[G3]
[subtotal 7]

- (iii) Convincing algebra to given result.

[M1]

r	0	1	2	3	4	5
x_r	2	2.242631	2.277768	2.278844	2.278862	2.278863

Root appears to be 2.27886 to 5 dp

[M1A1A1]
[A1]

x	$\sin x + \frac{2}{3}x$	
2.278855	<	2.2788625
2.278865	>	2.2788627

hence result is correct to 5 dp

[M1A1E1]
[subtotal 8]

[TOTAL 18]

4776 Numerical Methods (Written Paper)

General Comments

There was a lot of good work seen on this occasion, with few candidates appearing to be unready for the examination. Routine numerical calculations were generally carried out accurately, though it is yet again disappointing that so many candidates set their work out badly. This is an algorithmic subject and good work will reflect that. A poor layout is difficult to follow for the examiner – and difficult for the candidate to check.

Interpretation remains a weak area, with quite a number of candidates simply omitting such parts of questions. In particular, the requests for graphical explanations were often poorly done or not attempted at all.

Comments on individual questions

1 Bisection method to solve an equation

This proved a straightforward starter for most candidates. The best solutions set the method out in tabular form and kept careful track of the maximum possible error. Weaker solutions were more like a jumble of figures. The only errors of substance were failing to stop the process at the required point and counting the number of iterations incorrectly.

2 Numerical integration

This too proved straightforward for most candidates, with the equations linking the various integration rules being well understood. The accuracy of the final answer was sometimes misjudged, but most candidates got this right.

3 Error in $f(x)$

This appeared to be a less familiar area of the specification. The given result was generally established correctly, perhaps because it involves very simple algebra and calculus, but there were relatively few successful attempts to apply it. A few candidates answered part (ii) with no evidence that they had used the result in part (i). Given the explicit instruction “Hence ...”, it was not possible to credit these attempts.

4 Numerical error in computation

Again, the given result was obtained by almost all candidates. A smaller number, but still a majority, were able to show that the algebraic result and its numerical equivalent agreed for small k , but disagreed when k became large. The explanations of why this should be were often missing or poor. The two points being looked for were that computers and calculators do not carry out calculations exactly. More particularly the subtraction of nearly equal quantities, as in this case, is a common source of error.

5 Newton's forward difference formula

The difference table and the deduction from it proved straightforward. Most candidates were able to write down the appropriate cubic, but the simplification was often done poorly. Those who constructed a cubic in x had far more work to do than those who saw that they could take $x = 1.5$ from the outset.

6 Numerical differentiation

The graphical demonstration in part (i) that the central difference method is generally more accurate than the forward difference method was often poorly done. Very many candidates did *not* show the tangent to the curve; as a result they had nothing to compare the central and forward chords with.

Parts (ii) and (iii), however, were usually very well done. The numerical results were obtained accurately and the correct inference was drawn from the ratios of differences of estimates.

7 Numerical solution by fixed point iteration

The sketches in part (i) varied greatly in quality. The simplest approach was to sketch $y = x$ and $y = 3 \sin x$. Candidates were divided as to whether there were any other non-zero roots.

The numerical work in part (ii) was done well, but the cobweb diagrams were frequently poor. Even those that were essentially correct mostly omitted vital information about the direction to be taken around the spiral.

In part (iii) the algebraic rearrangement defeated a surprising number. The numerical work, once again, was done well.