Curriculum Overview: Critical Maths

Ninth draft

Background

Relative to its international competitors, England has very low rates of participation in post-16 mathematics[1] and Michael Gove has expressed the ambition that *"we should set a new goal for the education system so that within a decade the vast majority of pupils are studying maths right through to the age of 18."*[2]

In order to achieve this ambition, a new curriculum is needed, suitable for all students with a grade C or above in GCSE-level Mathematics. Professor Sir Timothy Gowers has some very interesting ideas on the nature of such a curriculum, as explained in his weblog, http://gowers.wordpress.com/2012/06/08/how-should-mathematics-be-taught-to-non-mathematicians/. Key features of Professor Gowers' ideas are as follows.

- Students should engage in solving real problems which will capture their imagination.
- All teaching and learning should stem from engaging with questions and enable students to develop quantitative literacy.
- In addition to enabling students to develop skills of analysing new situations and thinking flexibly in problem solving, the questions will stimulate the need for mathematical and statistical analysis.

The need for more students to study more mathematics

The Advisory Committee on Mathematics Education (ACME) argues that the motivation for increasing the number of students studying mathematics beyond the age of 16 goes beyond the utilitarian.

> "The utilitarian argument for increasing post-16 participation in mathematics is strong, but the case is bolstered further by considering mathematical study as a cultural activity – as an end in itself, to which a much larger proportion of the population should be exposed as a matter of course in a developed society."[3]

There has been much talk about Big Data recently; the availability of large amounts of data and the need for informed citizens to understand the implications of such data was identified in a 2001 United States National Council on Education and the Disciplines publication.

> "Working nonstop and with extraordinary speed, computers meticulously and relentlessly note details about the world around them and carefully record these details. As a result, they create data in increasing amounts every time a purchase is made, a poll is taken, a disease is diagnosed, or a satellite passes over a section of terrain. In consequence, one observer notes, whereas until very recently information about the world has been scarce and hard to come by, "today we are drowning in data, and there is unimaginably more on the way" (Bailey, 1996).
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> The implications of this new situation can be either very good or very bad. At present, they are some of both. For those competent and comfortable in thinking with numbers, the opportunities that come with the new conditions can be liberating. Not

[1] 'Is the UK an Outlier?', Nuffield Foundation, 2010
[2] Speech at the Royal Society, June 2011
[3] Post-16 Mathematics: A strategy for improving provision and participation, ACME 2012

> only specialists but now everyone can obtain and consider data about the risks of medication, voting patterns in a locality, projections for the federal budget surplus, and an almost endless array of other concerns. Potentially, if put to good use, this unprecedented access to numerical information promises to place more power in the hands of individuals and serve as a stimulus to democratic discourse and civic decision making. Indeed, as a recent study illustrates, the availability of numbers now reaches into "every nook and cranny of American life," making it no exaggeration to say that, in consequence, numerical thinking has become essential to "the discourse of public life" (Caplow et al., 2001). It follows, however, that if individuals lack the ability to think numerically they cannot participate fully in civic life, thereby bringing into question the very basis of government of, by, and for the people."[4]

One of the aims of Critical Maths is that this curriculum should encourage students to develop the knowledge, skills and understanding in mathematics and statistics they need to become educated citizens in the context of today's society. This includes an ability to draw conclusions from data and critically evaluate conclusions which are put forward by others. In addition to using the resources which will be developed by MEI, it is hoped that teachers will make use of suitable news stories with their students.

Key features of the teaching approach

Teachers will guide student discussion about the problems in a way that ensures that students think about the relevant features of the problem and are guided towards thinking of a solution for themselves. Professor Gowers describes the Socratic method and exemplifies how this might work in practice in his weblog[5]. It is important for the success of this approach that students are genuinely interested in the problems presented to them and are able to start discussing them, perhaps by expressing an opinion. Students' interest in finding the solution to the problem will motivate them to engage with the mathematics needed. Professor Gowers stresses the importance of this.

> "The main point is one I've basically made already: the discussions should *start from the real-life problem* rather than starting from the mathematics. Pupils should not feel that the question is an excuse to force some mathematics on them: they should be interested in the question and should *feel the need* for the mathematics, the need arising because one can give much better answers if one models the situation mathematically and analyses the model."[6]

In a study and research review of factors that support and inhibit mathematical reasoning, Henningsen and Stein found as follows.

> "Research has suggested that tasks that are likely to maintain high-level cognitive demands are tasks that build on students' prior knowledge (Bennett & Desforges, 1988) and are allotted an appropriate amount of time for the students to engage at a high level, that is, neither too little nor too much time (Doyle, 1986). Teaching behaviors that were found to support high-level student engagement in this study, including scaffolding, modelling high-level performance, and consistently pressing students to provide meaningful explanations, have also been identified by other

[4] Mathematics and Democracy: the case for quantitative literacy, Steen et al, National Council on Education and the Disciplines, 2001

[5] http://gowers.wordpress.com/2012/06/08/how-should-mathematics-be-taught-to-non-mathematicians/

[6] ibid

researchers as important influences in tasks that encourage students to engage at high levels (Anderson, 1989; Doyle, 1988)."[7]

Appropriate teaching guidance and professional development will be provided for teachers using the Critical Maths materials to ensure that they are able to use them successfully to engage students in thinking mathematically. The guidance and professional development will draw on successful approaches used in other countries such as the Japanese practice of neriage[8] and the Dutch practice of guided reinvention[9] in order to exemplify successful teaching strategies.

Curriculum name

We intend that this curriculum should be accessible to students who have achieved grade C (or above) in GCSE Mathematics; the name "Critical Maths" reflects our purpose that this curriculum should enable students to develop skills of thinking flexibly and creatively and reflecting on their own and other people's solutions to problems.

The term "problem solving" in mathematics has more than one possible meaning; the types of problem solving included in this curriculum are as follows.

- ✓ The act of thinking creatively about a non-routine question.
- ✓ Using mathematics to model and solve a real-life problem.
- ✓ Using mathematical ideas to solve an unfamiliar question type.

By contrast, the following interpretations of problem solving are **not** included in this curriculum.

- × Questions set in realistic contexts merely in order that students practise the mathematics they have learnt.
- × Providing a generic toolkit of strategies independently of addressing real problems.
- × Questions which are only appropriate for the most able students who intend to pursue a future career involving a good deal of mathematics.

[7] Mathematical Tasks and Student Cognition: Classroom-Based Factors That Support and Inhibit High-Level Mathematical Thinking and Reasoning, Henningsen and Stein, Journal for Research in Mathematics Education, Vol. 28, No. 5 (Nov., 1997)

[8] Neriage or "polishing up" involves extensive whole-class discussion and comparison of student solutions to problems in order to enable students to acquire new mathematical ideas. http://tsg.icme11.org/document/get/827

[9] In guided reinvention, the teacher has a clear solution path in mind and guides the students in such a way that they feel as though they constructed the knowledge for themselves. http://dare.ubvu.vu.nl/bitstream/handle/1871/10770/JCSGravemeijer&Terwel2000.pdf;jsessionid=8E6647FF82B7B649ACF0CD94AFE3F24D?sequence=1

ACME's work on post-16 Mathematics
The DfE has recently invited the Advisory Committee on Mathematics Education (ACME) to convene an expert panel on Core Mathematics qualifications to produce guidelines on the development of new qualifications are intended for students who have at least grade C in GCSE Mathematics but are not taking A level Mathematics.

ACME has previously produced recommendations about such qualifications, including the following.

> **"A new qualification should be developed and introduced as part of wider A level reforms.** This qualification should:
> - o Be distinct from A level Mathematics, with an emphasis on solving realistic problems, using a variety of mathematical approaches, and should be for students not currently doing AS or A level Mathematics
> - o Give students the confidence to consolidate their understanding of mathematics by using and applying mathematics already learned in GCSE and new mathematics beyond GCSE developed during the course.
> - o Have a smaller volume than AS level and be designed to be studied over two years."[10]

The Critical Maths curriculum meets the first two of these requirements and can be used to develop qualifications which would also meet the third.

In addition to meeting the first of ACME's requirements, Critical Maths would also be useful for students who are doing AS or A level Mathematics. Discussion with Professor Peter Main, director of education and science at the Institute of Physics, confirms that the kinds of skills developed in a Critical Maths course are important skills for physicists to develop; important skills identified included the following.

- The ability to explore situations by not being afraid to experiment. Considering extremes when testing solutions.
- The ability to produce Fermi estimates.
- An understanding of risk and the effect on public perception of a single high profile event and media attention.
- An understanding of exactly what quoted statistics mean.

The Critical Maths curriculum should increase students' confidence and help develop and consolidate their mathematical understanding. Crucially, such understanding is developed as a consequence of, not a pre-requisite for, consideration of realistic problems.

The Critical Maths curriculum could be used as the basis for qualifications which are studied over two years and are smaller in volume than an AS level. In order for the curriculum to be successful in achieving its aims for the students studying it, it is crucial that the assessment of such qualifications is in sympathy with these aims. This is by no means easy to achieve; there is a separate report on approaches to assessment which explores the issues associated with assessment and suggests some ways forward.

[10] Post-16 Mathematics: A strategy for improving provision and participation, ACME 2012

Appendix
Critical Maths content outline

Assumed knowledge

It is assumed that students can use and apply the content of the Foundation tier of GCSE Mathematics.

Aims

This curriculum should encourage students to

- Engage in solving realistic problems appropriate to this level.
- Recognise when mathematical and statistical analysis will be helpful.
- Develop skills of representing new situations mathematically and thinking flexibly in problem solving.
- Develop the ability to use their mathematical and statistical knowledge to make logical and reasoned decisions and communicate them clearly.
- Develop the mathematical and statistical knowledge and skills they need to become educated citizens in the context of today's society.
- Have the confidence to work on a problem where the method of solution is not obvious.

Objectives

Students should be able to

- Discuss problems, identifying the important features.
- Propose solutions to problems.
- Evaluate strategies for tackling a problem.
- Use quantitative evidence.
- Make reasonable estimates with limited information.
- Communicate their solutions, strategies and reasoning to others.
- Check a solution to see whether it is reasonable and criticise unreasonable solutions.
- Interpret mathematical solutions in terms of the original problem.
- Recognise related problems and apply the knowledge and skills they have learnt to real situations.

Question types

Problems addressed during the course should be characterised by the following features.

- Be set in realistic contexts.
- Encourage students to develop an intuitive understanding of key mathematical ideas.
- Do not have an obvious means of solution but require identification of underlying mathematics leading towards the solution.
- Where appropriate, include questions that have definite yes/no answers.
- Emphasise the importance of justification of answers.
- Have solutions which can be communicated in a way which can be understood by the intelligent citizen, without the use of specialised mathematical techniques.
- Use estimation and reasoned argument rather than exact calculation.

Competence statements

Understanding information

A student should:

- U1. Recognise what information is necessary in order to address a problem.
- U2. Extract relevant information from written sources.
- U3. Be able to find, read and comprehend information from a variety of graphs and tables.
- U4. Be able to use data from a variety of public sources including national statistics.

Problem solving

A student should:

- P1. Recognise the power and limitations of mathematical thinking to address a variety of problems.
- P2. Understand how to make simplifying assumptions to enable a problem to be solved.
- P3. Understand how to design suitable experiments to enable reasoned and informed choices to be made.
- P4. Understand and explain some of the possible sources of error/bias when choosing a sample, collecting data or conducting an experiment. For example, sample size being too small for confident conclusions, selection method or survey questions being biased.
- P5. Evaluate different strategies as part of a process of making decisions.
- P6. Make fair decisions and justify them.
- P7. Communicate problem solving strategies and interpret solutions effectively in terms of the original problem.
- P8. Recognise when a problem is related to one that they have already worked on.

Reflecting on solutions

A student should:

- R1. Interpret a solution in terms of the original problem.
- R2. Evaluate assumptions made in public statements such as news reports and political comments.
- R3. Evaluate the accuracy of results obtained, and balance this with choices made regarding time, cost and other factors when deciding on a model or experiment.
- R4. Recognise or formulate a problem related to one that they have already worked on.
- R5. Criticise or refine a proposed solution to a problem.

Types of problems which students should be able to solve

A student should:

- T1. Be able to produce Fermi estimates, justifying the choices made (see notes below).
- T2. Be able to estimate probabilities, including using Fermi estimation.
- T3. Discuss optimisation problems.
- T4. Make financial, business, and personal choices based on sound evidence, giving consideration to alternative models.
- T5. Be able to compare two quantities, by means of logical argument, without working out the value of either.
- T6. Be able to use extreme values to decide whether general statements are true or false.

Notes on estimation

Students are expected to have a quantitative understanding of the world around them including common measures of distance, mass and time. When producing estimates, students may be expected to have developed a reasonable understanding of the (approximate) size of the following quantities.

- a. The total population of the UK and of the world.
- b. The population of London.
- c. The average life expectancy in the UK.
- d. The number of children in a typical family in the UK.
- e. A typical wage in the UK.
- f. The weight of a typical suitcase for a holiday maker.
- g. The number of people which a typical car, bus, train or plane can seat.

Students may be given a data sheet in the examination which will enable them to make and use reasonable estimates of quantities. Students may be asked to make estimates of quantities such as the following without any further given information.

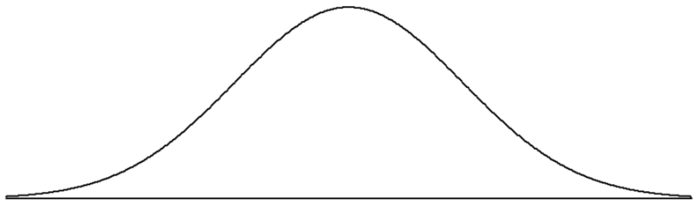
- a. The height of a typical adult, and other typical body measurements.
- b. The walking speed of a typical adult.
- c. The length of stride of a typical adult.
- d. The height of a typical step in a staircase.
- e. The size of a typical house or school in the UK.
- f. The number of cars in the UK.
- g. The number of hairs on a typical human head.
- h. The thickness of paper.
- i. The volume of rubbish discarded in a typical year by a typical UK household.
- h. The number of words a typical person can say in one minute.
- i. The volume of water drunk per day by a typical person.
- j. The volume of a typical cup of coffee.

Mathematical ideas which students should encounter through discussion of problems

A student should:

- M1. Understand the concepts behind risk and the communication of risk, including the benefits of using representative frequency in preference to fractions, decimals, percentages or ratio.
- M2. Know that the long-run relative frequency of an event approximates to the probability.
- M3. Be able to estimate an expected return from a game or investment.
- M4. Understand that an event with small probability is not surprising in a sufficiently large population.
- M5. Recognise fallacies in probability, such as the prosecutor's fallacy.
- M6. Recognise ways in which statistics are misused and be aware of common misconceptions.
- M7. Understand that correlation does not imply causation.
- M8. Understand the importance of randomised controlled trials when trying to establish causation.
- M9. Understand the importance of "blind" and "double blind" in statistical trials.
- M10. Recognise the shape of the Normal distribution and know that it can approximate the distribution of the sum of the outcomes of a large number of random events.
- M11. Know that, for an experiment which can be modelled by the toss of a fair coin, for n repetitions of the experiment, on average $n/2$ occurrences will happen and that values more than $\sqrt{n}$ away from this average are fairly unusual; values more than $2\sqrt{n}$ away from this average are very unusual.
- M12. Understand that proportions from small samples are more variable than from large samples.
- M13. Understand that natural variability can result in values that are above (or below) average.
- M14. Recognise regression to the mean.
- M15. Be able to work with large and small numbers and use standard form.
- M16. Recognise the characteristics of exponential growth and decay, including the idea that the time to double (or halve) the size is constant.
- M17. Understand scale factors in one, two and three dimensions and the relationships between them.
- M18. Know that two quantities being compared should have the same units.
- M19. Know that percentage change is an important way of comparing quantities.
- M20. Understand how the units of quantities arise from the way they were calculated e.g. metres per second arises from dividing a distance in metres by a time in seconds.
- M21. Recognise that, when trying to maximise some benefit, small changes a long way from the optimum typically result in noticeable changes in benefit but small changes close to the optimum result in small changes in benefit. Similarly for minimisation.

Glossary of terms which students are expected to know and be able to use

Association	A tendency for two events to occur together.
Control group	One of two groups in a randomised controlled trial. The control group does not receive the treatment being tested in the experiment.
Correlation	An association between two variables which is approximately linear.
Experiment	A deliberate action undertaken with the aim of observing the results.
Fermi estimate	A rough estimate using simplifying assumptions and approximate numbers to get an idea of the order of magnitude of a quantity.
Model	A simplified version of a real situation which is useful in solving a problem.
Normal distribution	A symmetrical continuous distribution which has a particular kind of bell shape. 
Order of magnitude	The nearest power of 10 to a number e.g. millions, hundreds etc.
Population	All the items or individuals of interest.
Prosecutor's fallacy	A probability error in which a probability of one event given that another is true is confused with the reverse. For example, about one in ten of the population is left handed; slightly more men than women are left handed. The probability of a man being left handed is about 12% but the probability of a left hander being a man is just over 50%.
Randomised controlled trial	A type of experiment where individuals are allocated to either the experimental group or the control group with pre-assigned probability.
Regression to the mean	Outcomes of an experiment which are a long way from the mean tend to be less extreme when the experiment is repeated. E.g. runners who run very fast in a race are more likely to be slower than faster next time.
Relative frequency	The frequency of an outcome out of the total number of experiments. Relative frequency is used as an estimate of probability.
Sample	A set of items chosen from a population.
Standard form	Expressing a number as a number between 1 and 10 times a power of 10. E.g. $4200 = 4.2 \times 10^3$.
Variability	The idea that there is a natural tendency for measurements of the same quantity in a sample or population to be spread over a range.