

Questions for STEP Section 01 – Division by zero

STEP
Mathematics I
Summer 2003

- 2 The first question on an examination paper is:

$$\text{Solve for } x \text{ the equation } \frac{1}{x} = \frac{1}{a} + \frac{1}{b}.$$

where (in the question) a and b are given non-zero real numbers. One candidate writes $x = a + b$ as the solution. Show that there are no values of a and b for which this will give the correct answer.

The next question on the examination paper is:

$$\text{Solve for } x \text{ the equation } \frac{1}{x} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$

where (in the question) a , b and c are given non-zero numbers. The candidate uses the same technique, giving the answer as $x = a + b + c$. Show that the candidate's answer will be correct if and only if a , b and c satisfy at least one of the equations $a + b = 0$, $b + c = 0$ or $c + a = 0$.

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- 6 Evaluate the following integrals, in the different cases that arise according to the value of the positive constant a :

(i)
$$\int_0^1 \frac{1}{x^2 + (a+2)x + 2a} dx;$$

(ii)
$$\int_1^2 \frac{1}{u^2 + au + a - 1} du.$$

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- 4 Show that $x^3 - 3abc + b^3 + c^3$ can be written in the form $(x + b + c)Q(x)$, where $Q(x)$ is a quadratic expression. Show that $2Q(x)$ can be written as the sum of three expressions, each of which is a perfect square.

It is given that the equations $ay^2 + by + c = 0$ and $by^2 + cy + a = 0$ have a common root k . The coefficients a , b and c are real, a and b are both non-zero, and $ac \neq b^2$. Show that

$$(ac - b^2)k = bc - a^2$$

and determine a similar expression involving k^2 . Hence show that

$$(ac - b^2)(ab - c^2) = (bc - a^2)^2$$

and that $a^3 - 3abc + b^3 + c^3 = 0$. Deduce that either $k = 1$ or the two equations are identical.

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- 1 It is given that $\sum_{r=-1}^n r^2$ can be written in the form $pn^3 + qn^2 + rn + s$, where p, q, r and s are numbers. By setting $n = -1, 0, 1$ and 2 , obtain four equations that must be satisfied by p, q, r and s and hence show that

$$\sum_{r=0}^n r^2 = \frac{1}{6}n(n+1)(2n+1).$$

Given that $\sum_{r=-2}^n r^3$ can be written in the form $an^4 + bn^3 + cn^2 + dn + e$, show similarly that

$$\sum_{r=0}^n r^3 = \frac{1}{4}n^2(n+1)^2.$$

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Mathematics
II Summer
2004**

- 1 Find all real values of x that satisfy:
- (i) $\sqrt{3x^2 + 1} + \sqrt{x} - 2x - 1 = 0$;
 - (ii) $\sqrt{3x^2 + 1} - 2\sqrt{x} + x - 1 = 0$;
 - (iii) $\sqrt{3x^2 + 1} - 2\sqrt{x} - x + 1 = 0$.

**STEP
Mathematics
III Summer
2005**

- 3 Let $f(x) = x^2 + px + q$ and $g(x) = x^2 + rx + s$. Find an expression for $f(g(x))$ and hence find a necessary and sufficient condition on a, b and c for it to be possible to write the quartic expression $x^4 + ax^3 + bx^2 + cx + d$ in the form $f(g(x))$, for some choice of values of p, q, r and s .

Show further that this condition holds if and only if it is possible to write the quartic expression $x^4 + ax^3 + bx^2 + cx + d$ in the form $(x^2 + vx + w)^2 - k$, for some choice of values of v, w and k .

Find the roots of the quartic equation $x^4 - 4x^3 + 10x^2 - 12x + 4 = 0$.