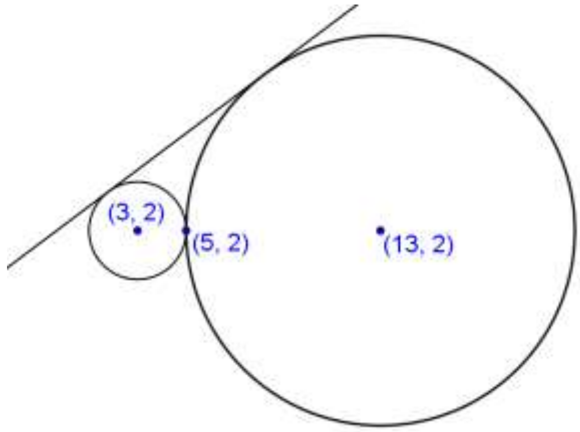
## Example 11



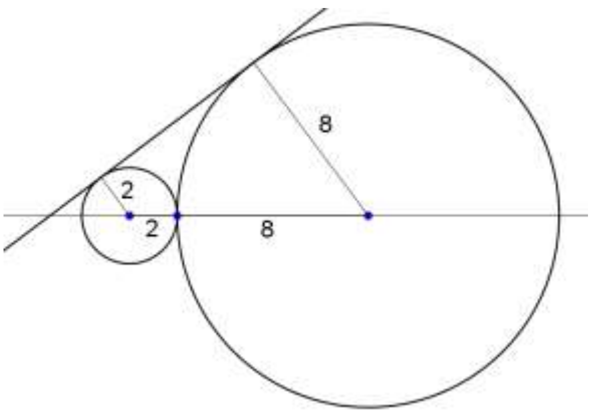

The diagram shows two circles and a straight line that is a tangent to both circles.

Find the equation of the line.

### Student 1

The radius of the small circle is 2 units and the radius of the large circle is 8 units.

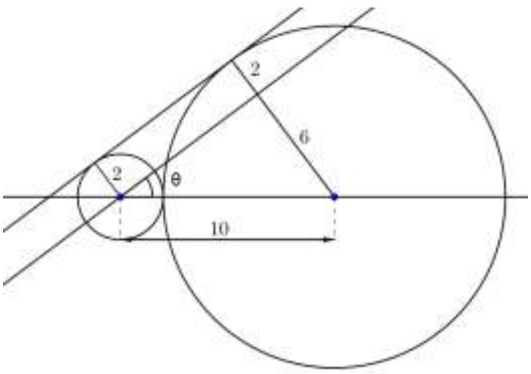
Drawing a diagram and putting these values on it




Two of the marked radii are parallel as the angle they make with the tangent is 90°.

Drawing a line through the centre of the circle parallel to the marked tangent gives

### Student 2



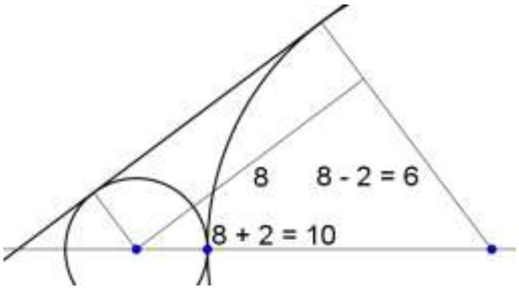

A line parallel to the tangent line is drawn through the centre of the smaller circle. The student realises that this gives a 6, 8, 10 triangle.

The student knows that $\tan\theta$ is the gradient and from their triangle $\tan\theta = \frac{3}{4}$

The equation of the small circle is

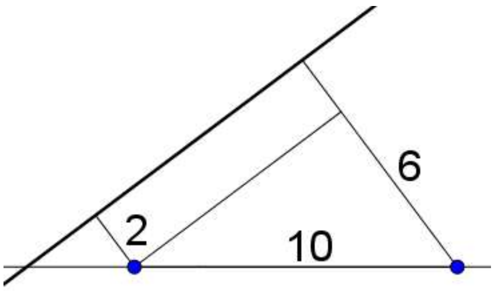
$$(x-3)^2 + (y-2)^2 = 4$$

## Student 1




We have a 3, 4, 5 triangle doubled to be 6, 8, 10.

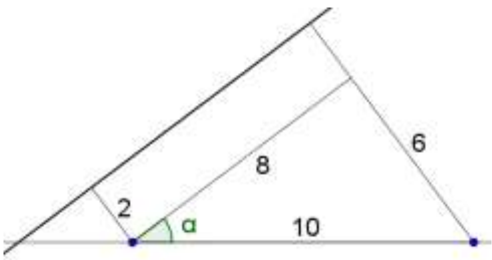
The small triangle on the left is similar to this




With all lengths $\frac{1}{3}$ of the larger triangle.

So the tangent passes through a point that is a horizontal distance of $\frac{10}{3}$ away from the point $(3, 2)$ i.e. the point $\left(-\frac{1}{3}, 2\right)$.

The gradient of the line can be found using one of the right angled triangles




$\tan\alpha = \frac{6}{8} = \frac{3}{4}$ so the gradient of the line is $\frac{3}{4}$

The equation of the line is $y - 2 = \frac{3}{4}\left(x + \frac{1}{3}\right)$

$4y - 8 = 3x + 1$

## Student 2

The equation of the large circle is

$(x - 13)^2 + (y - 2)^2 = 64$

Let the equation of the tangent be $y = \frac{3}{4}x + c$

where $m > 0$

Intersects with the small circle when

$$(x - 3)^2 + \left(\frac{3}{4}x + c - 2\right)^2 = 4$$

As it is a tangent this will only have one solution.

$$(x - 3)^2 + \left(\frac{3}{4}x + (c - 2)\right)^2 = 4$$

$$x^2 - 6x + 9 + \frac{9}{16}x^2 + \frac{3}{2}x(c - 2) + (c - 2)^2 = 4$$

$$\frac{25}{16}x^2 + \left(\frac{3}{2}c - 3 - 6\right)x + (c - 2)^2 + 5 = 0$$

$$\frac{25}{16}x^2 + \left(\frac{3}{2}c - 9\right)x + (c - 2)^2 + 5 = 0$$

For one solution the discriminant = 0

$$\left(\frac{3}{2}c - 9\right)^2 - 4 \cdot \frac{25}{16}((c - 2)^2 + 5) = 0$$

$$\frac{9}{4}c^2 - 27c + 81 - \frac{25}{4}(c^2 - 4c + 4 + 5) = 0$$

$$9c^2 - 108c + 324 - 25(c^2 - 4c + 9) = 0$$

$$9c^2 - 108c + 324 - 25c^2 + 100c - 225 = 0$$

$$-16c^2 - 8c + 99 = 0$$

$$16c^2 + 8c + 99 = 0$$

$$(4c - 9)(4c + 11) = 0$$

$c = \frac{9}{4}$ or $-\frac{11}{4}$

| **Student 1** | **Student 2** |
|---|---|
| $3x - 4y + 9 = 0$ | This student then surmised that $c = \frac{9}{4}$ “because it looks like it should be above the $x$ axis”<br>So $y = \frac{3}{4}x + \frac{9}{4}$<br>$4y = 3x + 9$<br>$3x - 4y + 9 = 0$ |

In this example, although student 2 made a good start and worked through some tortuous algebra to come up with the correct answer, their solution is nowhere near as elegant as that of student 1. It is clear that, in problem solving terms, student 1 has the better approach and has thought things through. Student 2 would need to go back and look to see where they should have stopped and thought about the effort they were having to go to with the algebra and concluded that they should have been looking for something more straightforward.