Mathematical Problem Solving
AS/A Level example

Example 5

An able class have been studying the coordinate geometry of the circle and have covered equations of circles and their properties.

Understand and use the coordinate geometry of the circle including using the equation of a circle in the form $(x-a)^2+(y-b)^2=r^2$; completing the square to find the centre and radius of a circle; use of the following properties:

- the angle in a semicircle is a right angle
- the perpendicular from the centre to a chord bisects the chord
- the radius of a circle at a given point on its circumference is perpendicular to the tangent to the circle at that point

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The teacher presents the class with this mystery design task either providing instructions on the board, as a worksheet or talking the students through the process. The students are placed into groups of 3, 4 or 5 and each group is provided with a set of the cards, some lined paper and a sheet of A3 or flipchart paper.

Student instruction sheet

You are going to find a design that has been created using circles.

Some information about the design is given on the cards.

- Shuffle the cards and deal them all out
- Take it in turns to select one piece of information that you think is important and read it to the rest of the group
- You can write something down on the group's answer sheet but you must not show anyone any of your cards (even after you have read them out)
- You can remind the group of any information you have already read out at any stage
- You will need to go round the group more than once to get enough information to sketch the design
- As a group, sketch the design labelling each circle with its equation

What is the fewest number of cards needed to solve the problem?

The student cards are on the next page.

The design consists of three circles.	The two smaller circles are contained within the largest circle.	All three circles have their centres on the positive x axis.
The two smaller circles overlap.	The largest circle passes through the point $(0,2)$.	The ratio of the radii of the two smaller circles is $2:3$
The smallest circle passes through the centre of the largest circle.	The second largest circle passes through the origin.	The second smallest circle passes through the centre of the smallest circle.
The centres of the two smaller circles are three units apart.	The largest circle passes through the point $(8,-2)$.	The design has reflection symmetry.
The areas of the three circles are in the ratio $4:9:20$	The area of the largest circle is 5 times the area of the smallest circle.	The point $(5,0)$ is inside all three circles.

Commentary

This commentary follows the discussions of a group of 5 students attempting to solve the problem. Their comments have been paraphrased to make them clearer for the reader.

The initial distribution of the cards amongst the 5 students was:

Student	Cards		
1	The second largest circle passes through the origin.	All three circles have their centres on the positive x axis.	The area of the largest circle is 5 times the area of the smallest circle.
2	The design consists of three circles.	The largest circle passes through the point $(0,2)$.	The design has reflection symmetry.
3	The centres of the two smaller circles are three units apart.	The two smaller circles overlap.	The point $(5,0)$ is inside all three circles.
4	The two smaller circles are contained within the largest circle.	The largest circle passes through the point $(8,-2)$.	The smallest circle passes through the centre of the largest circle.
5	The second smallest circle passes through the centre of the smallest circle.	The ratio of the radii of the two smaller circles is $2:3$	The areas of the three circles are in the ratio $4:9:20$

The students' comments and notations are detailed below. The solution to the problem can be found in appendix 1.

Student	Card played	Discussion and calculation	Output
1	All three circles have their centres on the positive x axis.	There are three circles. We can draw something	Draws an x axis and marks 0 near the left hand side. The x axis is fairly central to the paper despite there being no evidence that it should be! Writes 3 at the top of the page and circles it.
2	The largest circle passes through the point $(0,2)$.	We can put this on the diagram. How do we know it's the largest circle? Put an L by it.	Draws a y axis. Marks a 2 on it. Marks a X at $(0,2)$ and labels it L
3	The centres of the two smaller circles are three units apart.	We can't do a lot with this yet but I'm sure it will be useful later. I'll call the smaller circles M and S for medium and small	Draws a pair of smallish circles (overlapping) at the edge of the page with an arrow between the centres with 3 written on it. Labels one circle M and the other S. Note: 3 also has a card saying the two smaller circles overlap and has clearly taken this into account when noting the information at the edge of the page!
4	The largest circle passes through the point $(8,-2)$.	We've now got two points the big circle goes through. Can we find the whole circle from two points? <<General discussion about this>> No, we'll need at least one more point.	Marks a X at $(8,-2)$ and labels it L

Student	Card played	Discussion and calculation	Output
5	The second smallest circle passes through the centre of the smallest circle.	(To 3) We'll have to change your sketch a bit	Draws a new sketch of the overlapping circles below the one drawn previously. Marks them M and S and makes sure M goes through the centre of S
		(After the new drawing) The radius of S is less than 3. <<Sudden realisation>> The radius of M is 3	Writes $r = 3$ next to the sketch of M and S.
1	The second largest circle passes through the origin.	Does this help? <<General discussion>> The M and S are probably the other way round otherwise the centre of S won't be positive. If the centre is on the positive x axis, and it goes through $(0,0)$ then its centre must be here	Sketches the M and S diagram the other way round Marks a X at $(3,0)$ and draws a circle that goes through $(0,0)$ marks this as M

2	The design has reflection symmetry.	None of my cards has anything really useful on them. If all the centres are on the x axis we know that it must have at least one line of reflection symmetry anyway. Hold on, if the big circle goes through $(8,-2)$ and $(0,2)$, it must go through $(8,2)$ and $(0,-2)$ as well – it's not as bad a card as I thought!	Marks a half-hearted dotted line along the x axis.
		Can we get the centre of the big circle from this? <<General discussion>> If we draw a straight line from $(8,-2)$ to $(0,2)$ and from $(8,2)$ to $(0,-2)$, they should cross at the centre of the circle 'cos the y coordinates are the same. <<Sudden realisation>> That's just going to be at the half way point – the centre of the large circle is at $(4,0)$	Marks $(8,2)$ and $(0,-2)$ with a X and an L Sketches in two faint lines – one joining $(8,-2)$ to $(0,2)$ and the other joining $(8,2)$ to $(0,-2)$
		What's its radius? <<Paper calculation>> From $(4,0)$ to $(0,2)$ is $\sqrt{16+4}=\sqrt{20}$ Oh great, surds!	Marks a X at $(4,0)$ and sketches in the large circle. Writes $\sqrt{20}$ on the line joining $(4,0)$ to $(0,2)$.

Student	Card played	Discussion and calculation	Output
3	The point $(5,0)$ is inside all three circles.	I've got nothing left of use. <<Group considers this information for a bit>> No, it's not much use. Next.	Marks a X at $(5,0)$.
4	The smallest circle passes through the centre of the largest circle.	I reckon we can finish it with this. (3 interjects) My first card said the centres are 3 apart so the smallest circle must have a centre at $(6,0)$ If it goes through $(4,0)$ it must have a radius of 2. Is that it? Do you reckon we should check it. We usually have to so I suppose we should!	Marks a X at $(6,0)$ Sketches in a circle with radius 2

Student	Card played	Discussion and calculation	Output
5	The areas of the three circles are in the ratio $4 : 9 : 20$	I only got to read out one of mine – we could use this one to check <<Paper calculation>> $\pi \times 2^2 = 4\pi$ $\pi \times 3^2 = 9\pi$ $\pi \times \left(\sqrt{20}\right)^2 = 20\pi$ That looks right. We could do with drawing it again as it's really messy. What about the equations? <<Paper calculations>> $(6,0)$ and $r = 2$ $(x-6)^2 + y^2 = 4$ $(3,0)$ and $r = 3$ $(x-3)^2 + y^2 = 9$ $(4,0)$ and $r = \sqrt{20}$ $(x-4)^2 + y^2 = 20$ We only really used 9 cards – let's check the others.	Turns the sheet over and redraws the diagrams Writes the equations by the circles.
1	The area of the largest circle is 5 times the area of the smallest circle.	4π and 20π – that's OK	
2	The design consists of three circles.	Yeah, pointless card	

Student	Card played	Discussion and calculation	Output
3	The two smaller circles overlap.	They do – I knew this when I sketched them earlier (turns over paper and indicates original sketch)	
4	The two smaller circles are contained within the largest circle.	Yes they are!	
5	The ratio of the radii of the two smaller circles is $2:3$	They're 2 and 3 so, yes.	

The group had time to discuss the fewest number of cards needed and decided the ones they had chosen were fairly good with the exception of the one identifying the reflection symmetry and the one that said $(5,0)$ is inside all of the circles. They agreed that, even though 3 drew the two smaller circles overlapping, they would have been able to get the centres of the two smallest circles anyway. The consensus of the group was that the minimum number of cards required was 7. It was good to have some of the other cards to check the answer was correct.