Solution to example 19

A geometric progression has first term a and common ratio r, and the terms are all different. The first, second and fourth terms of the geometric progression form three consecutive terms of an arithmetic progression. Given that the geometric progression converges and has a sum to infinity of $3+\sqrt{5}$, find the exact values of a and r.

Let t_i denote the i^{th} term of the geometric progression. Since the first term is a ($\neq 0$) and the common ratio r, we have

$t_1 = a$

$t_2 = ar$

$t_4 = ar^3$

We are given that

$t_4 - t_2 = t_2 - t_1$ (the terms form an arithmetic progression).

Therefore

$ar^3 - ar = ar - a$

and so, dividing through by the (non-zero) a, we obtain

$r^3 - r = r - 1$

$r^3 - 2r + 1 = 0$

Since $r = 1$ is a solution of this equation, the Remainder Theorem tells us that $(r-1)$ is a factor of the left-hand side $(1 - 2 + 1 = 0)$

To find the quadratic factor we examine the identity

$(r-1)(r^2 + kr - 1) \equiv r^3 - 2r + 1$

Equating coefficients of r^2 gives $k - 1 = 0 \Rightarrow k = 1$

The equation therefore becomes

$(r-1)(r^2 + r - 1) = 0$

Either $r = 1$ which would make all of the terms the same since

$t_1 = a$

$t_2 = ar = a$

$t_3 = ar^2 = a$

$t_4 = ar^3 = a$

or

$r^2 + r - 1 = 0$

So $r = \frac{-1 \pm \sqrt{1+4}}{2} = \frac{-1 \pm \sqrt{5}}{2}$

$r = \frac{-1+\sqrt{5}}{2}$ since $|r| < 1$ for convergence

Using the sum to infinity

$S_\infty = 3 + \sqrt{5}$

Since $S_\infty = \frac{a}{1-r}$

$$\frac{a}{1 - \frac{-1+\sqrt{5}}{2}} = 3 + \sqrt{5}$$

$$a = \left(3 + \sqrt{5}\right)\left(\frac{2 + 1 - \sqrt{5}}{2}\right)$$

$$a = \frac{1}{2}\left(3 + \sqrt{5}\right)\left(3 - \sqrt{5}\right)$$

$$a = \frac{1}{2}(9 - 5) = 2$$