Mathematical Problem Solving
AS/A Level example

Solution to example 7

Problem A

You have been taught the following identities:

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}, \qquad A + B \neq k + \dfrac{\pi}{2}$

You should know the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$ for $\theta = 0°$, $30°$, $45°$ and $60°$ and the sine and cosine of $90°$.

Problem

Part (i)

Find an expression for $\sin 75°$ in the form $\frac{\sqrt{a}+\sqrt{b}}{c}$, where a, b and c are positive integers.

Part (ii)

Find the value of $\sin 75° - \cos 75°$.

Give your answer in surd form.

Problem B

You have been taught the following identities:

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}, \qquad A + B \neq k + \dfrac{\pi}{2}$

$\sin 2A = 2 \sin A \cos A$

$\cos 2A = \cos^2 A - \sin^2 A$

$\sin^2 A + \cos^2 A = 1$

You should know the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$ for $\theta = 0°$, $30°$, $45°$ and $60°$ and the sine and cosine of $90°$.

Problem

Part (i)

Find an expression for $\cos 3A$ in the form

$a\cos^n A + b\cos A$ where a, b and n are integers

Part (ii)

Given that $\cos 3x = 1$, and without finding the value of x, show that there are two possible values of $\cos x$ and find these values.

Solutions

Problem A

(i) $\sin 75 = \sin(30 + 45)$

$= \sin 30 \cos 45 + \cos 30 \sin 45$

$= \frac{1}{2} \cdot \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2}$

$= \frac{\sqrt{2} + \sqrt{6}}{4}$

(ii) $\cos 75 = \cos(30 + 45)$

$= \cos 30 \cos 45 - \sin 30 \sin 45$

$= \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} - \frac{1}{2} \cdot \frac{\sqrt{2}}{2}$

$= \frac{\sqrt{6} - \sqrt{2}}{4}$

$\sin 75 - \cos 75 = \frac{\sqrt{2} + \sqrt{6}}{4} - \frac{\sqrt{6} - \sqrt{2}}{4}$

$= \frac{\sqrt{2} + \sqrt{6} - \sqrt{6} + \sqrt{2}}{4}$

$= \frac{2\sqrt{2}}{4} = \frac{\sqrt{2}}{2}$

Problem B

(i) $\cos 3A = \cos(A + 2A)$

$= \cos A \cos 2A - \sin A \sin 2A$

$= \cos A\,(\cos^2 A - \sin^2 A) - \sin A\,(2 \sin A \cos A)$

$= \cos^3 A - \cos A \sin^2 A - 2 \cos A \sin^2 A$

$= \cos^3 A - 3 \cos A \sin^2 A$

$= \cos^3 A - 3 \cos A\,(1 - \cos^2 A)$

$= \cos^3 A - 3 \cos A + 3 \cos^3 A$

$= 4 \cos^3 A - 3 \cos A$

(ii) $\cos 3x = 1$

$4 \cos^3 x - 3 \cos x = 1$

Let $c = \cos x$

$4c^3 - 3c - 1 = 0$

$c = 1$ gives $4 - 3 - 1 = 0$ so $(c - 1)$ is a factor

$(c - 1)(4c^2 + kc + 1) \equiv 4c^3 - 3c - 1$

Coefficients of c^2: $k - 4 = 0 \Rightarrow k = 4$

$4c^3 - 3c - 1 = 0$ factorises to

$(c - 1)(4c^2 + 4c + 1) = 0$

$(c - 1)(2c + 1)^2 = 0$

This has two roots so there are two possible values of $\cos x$

These are 1 and $-\frac{1}{2}$