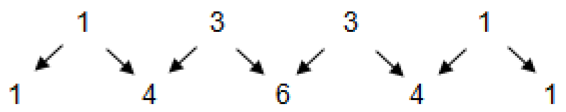
The Binomial Expansion (AS)

D1	[Understand and use the binomial expansion of $(a+bx)^n$ for positive integer *n*; the notations *n*! and *n*C*r*; link to binomial probabilities]

Commentary

Students often rely on Pascal's triangle to give the coefficients in a binomial expansion without understanding why this works. You might like to look at the connection between adding rows in Pascal's triangle:



and expanding brackets:

$$\begin{aligned}(1+x)^4 &= (1+x)^3(1+x)\\ &= 1+3x+3x^2+x^3 \longleftarrow (1+x)^3\times 1\\ &\quad\; +x+3x^2+3x^3+x^4 \longleftarrow (1+x)^3\times x\\ &= 1+4x+6x^2+4x^3+x^4\end{aligned}$$

In order to make the connection between the coefficient of x^m in the expansion of $(1+x)^n$ and ${}^nC_m$ (the number of ways of choosing m objects from n objects), students need to appreciate that when expanding brackets they are taking one element from each bracket and multiplying them together, and repeating this until they have done it in every possible way. When expanding $(1+x)^n$,choosing m of the n brackets from which to use the x, whilst using the 1 from each of the remaining $n-m$ brackets means there must be ${}^nC_m$ terms in the expansion which are equal to $x^m.1^{n-m}$.

Look for opportunities to link this to number work; for example, using the binomial expansion of $(-1+10)^9$ to deduce the tens and units digits in 9^9. Extend to 9^{99} and 99^{99}.

This situation is equivalent to the problem of determining the probability of m successes from n independent trials of an experiment with two possible outcomes. Choosing m of the n trials to be successes with probability p, whilst the remaining $n-m$ trials will be unsuccessful with probability $1-p$, shows there are ${}^nC_m$ mutually exclusive events with probability $p^m(1-p)^{n-m}$. This forms the basis of calculating probabilities using the binomial distribution.

Sample MEI resource

This ‘Binomial Expansion section test’ (which can be found at http://integralmaths.org/sow-resources.php) is from the Integral resources. There are similar tests for all topics and students can enter their answers on line for immediate feedback.

The binomial expansion

Section test

1) Find the value of ${}^{12}C_3$.

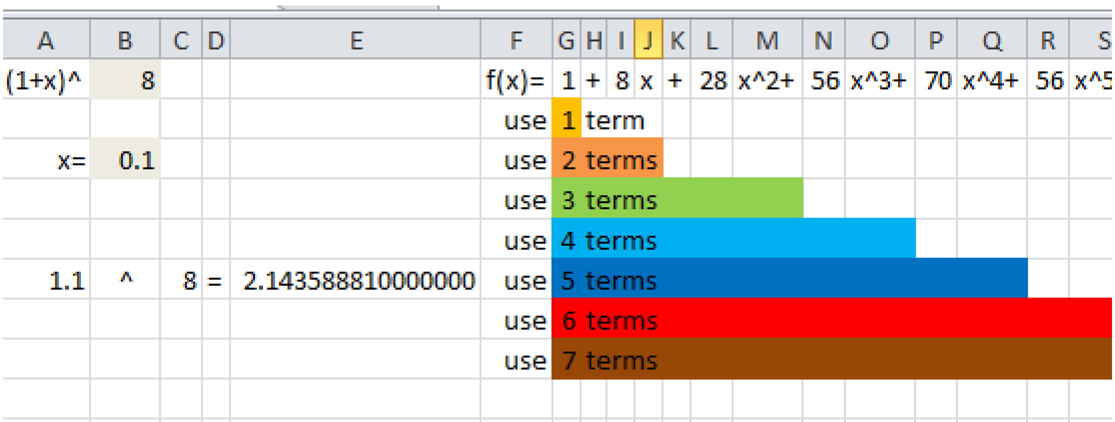
2) Find the value of ${}^{15}C_{11}$.

3) In the expansion of $(a-b)^5$, find the term in a^2 and the term in b^4.

4) In the expansion of $(x+3)^8$, find the coefficient of x^3, the coefficient of x^4 and the coefficient of x^6.

Effective use of technology

The ‘Binomial Expansion’ excel spreadsheet (found at www.mei.org.uk/integrating-technology) allows the user to investigate how using different numbers of ascending power of x terms in the binomial expansion of $(1+x)^n$ affects the accuracy of an estimate.



A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S
(1+x)^	8				f(x)=	1	+	8	x	+	28	x^2+	56	x^3+	70	x^4+	56	x^5
					use	1	term											
x=	0.1				use	2	terms											
					use	3	terms											
					use	4	terms											
1.1	^	8	=	2.143588810000000	use	5	terms											
					use	6	terms											
					use	7	terms											

How does the accuracy depend on x? Why? How does it depend on n?

# The Binomial Expansion	Time allocation:

Pre-requisites

- GCSE: Expanding brackets
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Links with other topics

- Surds and indices: for example, expanding $\left(1+\sqrt{2}\right)^n$
- Logarithms and exponentials: expanding $\left(1+\frac{1}{n}\right)^n$ for large values of n approaches a limit (the exponential number).
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Questions and prompts for mathematical thinking

- How would you explain why the coefficient of x^7 in $(2+x)^{10}$ is ${}^{10}\mathrm{C}_7 \times 2^3$?
- Change one number in $(1+1x)^4$ so that the coefficient of x in the expansion is 32.
- Give me two examples of binomial expansions in which all the coefficients are odd.
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Opportunities for proof

- Prove that ${}^nC_{r-1} + {}^nC_r = {}^{n+1}C_r$ (a property evident from Pascal's Triangle)
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Common errors

- Raising only part of the term to the appropriate power. For example, in $(1-2x)^6$, giving the third term as ${}^6C_3x^3$ or ${}^6C_3(2x)^3$ rather than ${}^6C_3(-2x)^3$
- Bracketing errors when evaluating a binomial coefficient e.g. giving the x^3 term in $(3-2x)^5$ as $10\times3^2\times2x^3$ or even $10\times3\times(-2)\times x^3$.
- Wasting time writing out the full expansion instead of finding the coefficient of the required term.
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