Two problems, with or without calculus

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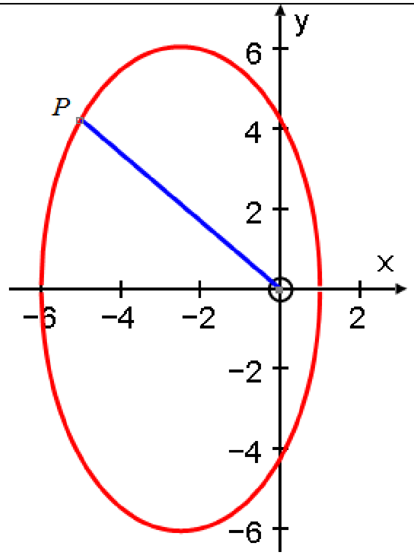
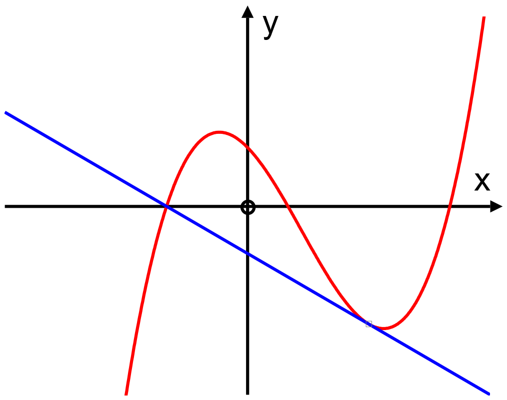
<tr><td>
</td><td>
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<tr><td>1. A point P moves along the curve which has equation $3x^2 + y^2 + 15x = 18$.

What is the greatest possible distance of P from the origin?</td><td>2. Take any cubic with three real roots and find the tangent at the average of two roots.

In the example shown, the tangent passes through the other root. Prove that this always happens.</td></tr>
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	With calculus	Without calculus
1	Let the square of the distance be L; i.e. $L=x^2+y^2$. But since $3x^2+y^2+15x=18$, it follows that $L=18-2x^2-15x$. Therefore $\frac{\mathrm{d}L}{\mathrm{d}x}=-4x-15$ and this has a stationary point when $x=-\frac{15}{4}$. (You can see this is a maximum since $\frac{\mathrm{d}^2L}{\mathrm{d}x^2}<0$). So the maximum distance is when P has coordinates $\left(-\frac{15}{4},\pm\frac{3\sqrt{57}}{4}\right)$. i.e. the distance is $\frac{3}{4}\sqrt{82}$	Solve simultaneously $3x^2+y^2+15x=18$ and $x^2+y^2=L$ knowing there is a repeated root. This gives $2x^2+15x-18+L=0$ and for repeated roots we need a discriminant of 0: $15^2=4\times2\times(L-18)\quad\Rightarrow\quad L=\frac{225}{8}+18=\frac{369}{8}$ giving a distance of $\sqrt{\frac{369}{8}}=\frac{3}{4}\sqrt{82}$
2	This is equivalent to proving the result after a horizontal translation, bringing the given midpoint to the y-axis. Then we have two roots symmetrically placed about the origin: $\mathrm{g}(x)=(x-\alpha)(x+\alpha)(x-\beta)$. Expanding brackets: $\mathrm{g}(x)=x^3-\beta x^2-\alpha^2x+\alpha^2\beta\quad\Rightarrow\quad\mathrm{g}'(x)=3x^2-2\beta x-\alpha^2$ The midpoint of the roots is now $x=0$ and so the equation of the tangent is $\frac{y-\mathrm{g}(0)}{x-0}=\mathrm{g}'(0)\quad\Rightarrow\quad\frac{y-\alpha^2\beta}{x}=-\alpha^2$ Substituting $y=0$ gives $x=\beta$, the third root.	Any line through the point $(c,0)$ has equation $y=m(x-c)$ (or $x=c$ but that one's of no interest in this problem). Consider where this line meets the cubic; i.e. $k(x-a)(x-b)(x-c)=m(x-c)$. Cancelling the common factor (the curve and line cross at $x=c$) we are left with the quadratic $k(x-a)(x-b)=m$ which expands to give $x^2-(a+b)x+ab-\frac{m}{k}=0$. For repeated roots we need this to be of the form $(x-\alpha)^2=0$ in which case $2\alpha=a+b$ and so the repeated root is $\frac{a+b}{2}$. Therefore this is the point whose tangent passes through $(c,0)$.

Alternative non-calculus approach to the second problem

If the equation of the cubic is $y = k(x-a)(x-b)(x-c)$ and the tangent at $x = \frac{a+b}{2}$ is $y = mx + q$ then the equation $k(x-a)(x-b)(x-c) = mx + q$ must have a repeated root of $x = \frac{a+b}{2}$ and a third root which we'd expect to be $x = c$ but we won't assume that yet. It follows that $k(x-a)(x-b)(x-c) - (mx+q) \equiv k(x-\alpha)\left(x - \frac{a+b}{2}\right)^2$

Comparing constant terms or coefficients of x would become messy (try it) and the coefficients of x^3 have been set up to be equal so compare coefficients of x^2: $-k(a+b+c) = k(-\alpha - (a+b)) \quad \Rightarrow \quad \alpha = c$ as required.