Exponentials and logarithms (AS)

F1	**[Know and use the function a^x and its graph, where a is positive]** **[Know and use the function e^x and its graph]**
F2	**[Know that the gradient of e^{kx} is equal to ke^{kx} and hence understand why the exponential model is suitable in many applications]**
F3	**[Know and use the definition of $\log_a x$ as the inverse of a^x, where a is positive and $x \geq 0$]** **[Know and use the function $\ln x$ and its graph]** **[Know and use $\ln x$ as the inverse function of e^x]**
F4	**[Understand and use the laws of logarithms:** $\log_a x + \log_a y = \log_a (xy)$; $\log_a x - \log_a y = \log_a \left(\frac{x}{y}\right)$; $k\log_a x = \log_a x^k$ **(including, for example, $k = -1$ and $k = -\frac{1}{2}$)]**
F5	**[Solve equations of the form $a^x = b$]**
F6	**[Use logarithmic graphs to estimate parameters in relationships of the form $y = ax^n$ and $y = kb^x$, given data for x and y]**
F7	**[Understand and use exponential growth and decay; use in modelling (examples may include the use of e in continuous compound interest, radioactive decay, drug concentration decay, exponential growth as a model for population growth); consideration of limitations and refinements of exponential models]**

Commentary

Many natural phenomena can be modelled by a function of the form ka^x and many students will be familiar with the term *exponential growth*. The graph of $y = ka^x$ can be plotted for some simple values of a and it can be observed that the value of a represents a growth factor. It can also be observed that the rate at which it is increasing at any point is proportional to its size at that point and hence the function and its gradient function have similar behaviour.

When working with functions such as $ax+b$, kx^2 and x^3, students have an intuitive grasp of what the inverse function is. Similarly trig and inverse trig functions can be understood by relating them to the unit circle. Functions of the form a^x are abstract: using a few numeric values of powers of 2 to draw the graph of the continuous function $y = 2^x$ ignores the issue of where other values come from. Consequently, it isn't surprising that students find the inverse - the logarithmic function - abstract and difficult. Furthermore, it is important to be aware of logarithms as a process ('find the log of 100') and as a function (which can be transformed, differentiated, and so on.)

Given a collection of points that lie on a curve of the form $y = ka^x$ it is not easy to find the corresponding values of k and a. However, taking logarithms of both sides shows that plotting $\log y$ against x, rather than y against x, would give a straight line from which it is straightforward to find the values of k and a. This important use of logarithms is covered in this unit.

Sample MEI resource

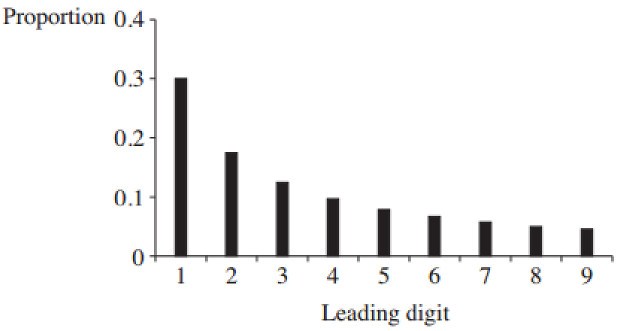
'Benford's Law' is from the January 2007 OCR(MEI) A level Core 4 comprehension paper (which can be found at http://integralmaths.org/sow-resources.php). It introduces students to an application of logarithms. The article is followed by questions which encourage the student to think about the underlying mathematical ideas.

Benford's Law

This phenomenon was noted in 1881 by Simon Newcombe, an American mathematician and astronomer, and then rediscovered by the physicist Frank Benford in 1938. Benford analysed 20 229 sets of data, including information about rivers, baseball statistics and all the numbers in an issue of *Reader's Digest*. He was rewarded for his efforts by having the law named after him.

Benford's Law gives a formula for the proportions of leading digits in data sets like these. This formula will be derived over the next few pages.

The proportions given by Benford's Law are illustrated in Fig. 8.



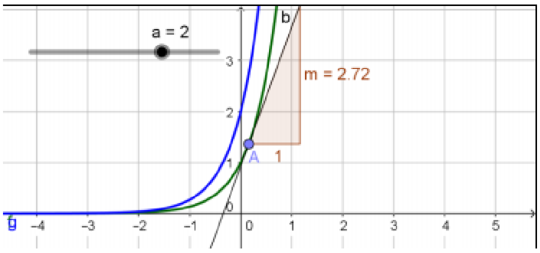
See http://www.mei.org.uk/alevelpapers for all the OCR(MEI) comprehension papers

Effective use of technology

'Derivative of exponential functions' (which can be found at http://www.mei.org.uk/integrating-technology) guides students through an exploration, using GeoGebra, into the gradient function of functions of the form e^{kx}.

Task 6: Derivate of exponential functions $y=e^{kx}$

1. Use the Slider tool to add a slider named **k**
2. Enter the function $f(x)=e^{kx}$: **f(x)=exp(k*x)**
3. Use the **Point** tool to add a point **A** on the function $f(x)=e^{kx}$
4. Use the **Tangent** tool to add a tangent to $f(x)=e^{kx}$ at **A**
5. Use the **Slope** tool to measure the gradient of the tangent, **m**
6. Enter the gradient function: **g(x)=f '(x)**



Exponentials and logarithms (AS)	**Time allocation:**

Pre-requisites

- Surds and Indices: the laws of indices lead directly to the laws of logarithms
- Differentiation: to appreciate the gradient function of $y = \mathrm{e}^{kx}$
- Graphs and transformations: to appreciate the link between $y = \mathrm{e}^{x}$ and $y = \mathrm{e}^{kx}$
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Links with other topics

- Geometric sequences: taking the logarithm of each term in a geometric sequence results in an arithmetic sequence
- Integration: $\int \frac{1}{x}\,\mathrm{d}x = \ln x + c$
- Differential equations: the general solution of $\frac{\mathrm{d}y}{\mathrm{d}x} = y$ is $y = A\mathrm{e}^{x}$
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Questions and prompts for mathematical thinking

- Is taking logs a useful technique for solving the equation $3^x = x^3$?
- Explain how can you use the fact that $2^{10} \approx 1000$ and $3^3 \approx 5^2$ to find approximate values of $\log_{10} 2, \log_{10} 5$ and $\log_{10} 3$?
- How can you use the value of $\log_{10} 2$ to determine the number of digits in the number 2^{74}?
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Opportunities for proof

- Starting from the laws of indices, prove the laws of logarithms
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Common errors

- Incorrectly stating the laws of logarithms; for example $\log_{10} a - \log_{10} b = \frac{\log_{10} a}{\log_{10} b}$
- Solving equations with products where logarithms need to be taken. E.g. $P = aT^n \quad \Rightarrow \quad \log P = \log a \times n \log T$ instead of $\log P = \log a + n \log T$
- After reducing to linear form, linking gradients and y-intercepts from their straight line to the original equation.
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