Ritangle 2025 Stage 3 Task

Consider the infinite *x*-*y* plane. Your task is to cover all the integer grid points on that plane (i.e. all the points (i,j) where $i \in \mathbb{Z}$, $j \in \mathbb{Z}$) with a repeating pattern of discs. A point on the edge of a disc is regarded as being covered. The discs may touch but not overlap. You may use discs of up to three different sizes. The objective is to use as few discs as possible, i.e. to maximise the mean number of points per disc, *P*. As a tie-break between solutions with the same *P*, the solution with the lowest area coverage *Q* (the fraction of the area of the plane that is covered by the discs) will be preferred.

You need to construct a **unit cell** for your solution. The unit cell must cover the plane using translations only; repeating identical copies of it must generate a set of whole discs that cover the grid points in the whole plane. If rotations of an original shape are required, attach the rotated shape to the original to form a unit cell that needs translations only.

For example, in Figure 1 below, the equilateral triangle ABD does not form a unit cell because a rotated copy is needed to tile the plane. However, the parallelogram (rhombus) ABCD is a candidate unit cell.

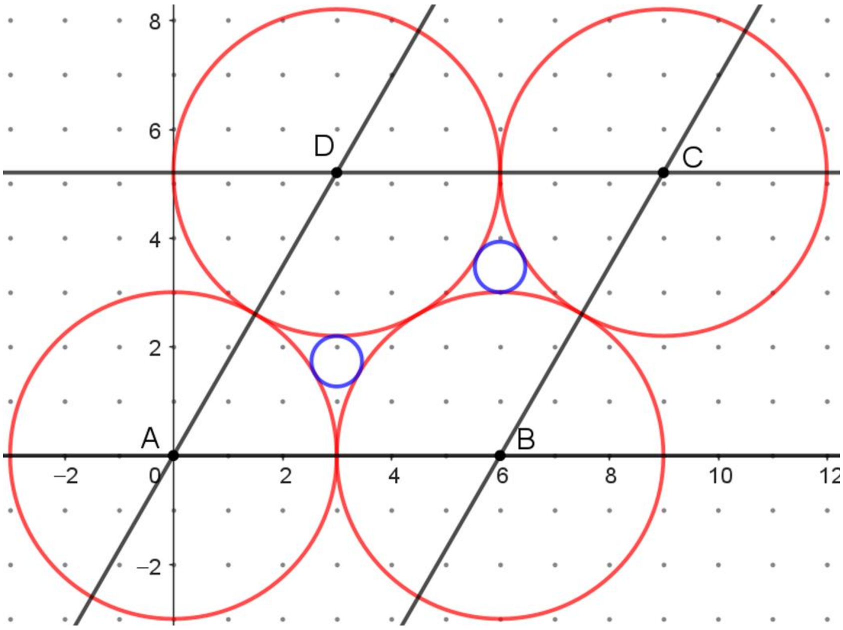

Figure 1 Candidate unit cell

However, there are a few issues with this 'solution':

- It uses only two sizes of disc (which is permitted, but is clearly not optimal).
- It does not cover the grid points (3, 1), (6, 4) or (6, 5) within the proposed unit cell.
- Repeated translations of the proposed unit cell to cover the whole plane will not generate the required integer grid.

Once you have found your solution, you should submit as comma-separated data, rounded to 5 decimal places where necessary:

- the number of vertices of the unit cell
- for each vertex, the (x,y) coordinates (each vertex on a new line, proceeding around the polygon either clockwise or anti-clockwise)
- the number of discs that appear (at least partially) in each unit cell
- for each disc, the (x,y) coordinates of the centre of the disc, followed by its radius and the fraction of the disc that lies within the unit cell (new line for each disc; the discs can be entered in any order)
- *P*, the mean number of points per disc in the unit cell and hence for the infinite repeating pattern
- *Q*, the fraction of the area of the unit cell that is covered by the discs
- an image of the unit cell, showing the grid

Calculate *P* for the unit cell by adding up the fractions of discs included in the unit cell to get a total number of discs. Count points on an edge as $\frac{1}{2}$, points on a vertex that is shared by three unit cells as $\frac{1}{3}$, points on a 4-way vertex as $\frac{1}{4}$, etc. For example, the candidate unit cell ABCD in the figure above contains:

- two quarter-points at A and B
- five half-points between A and B
- 30 other integer grid points
- one-sixth of a red disc centred at each of A and C
- one-third of a red disc centred at each of B and D
- two green discs

Therefore, the mean number of points per disc in the proposed unit cell is 33 / 3 = 11.

The area of the rhombus is $6 \times 3\sqrt{3}$. The large discs have radius 3 and the small discs radius $2\sqrt{3} - 3$. The unit cell contains the equivalent of one large disc and two small ones, so the fractional coverage is

$$Q = \frac{\pi.3^2 + 2.\pi\left(2\sqrt{3} - 3\right)^2}{6 \times 3\sqrt{3}}.$$

The data for the candidate unit cell and discs in the example above would be:

```
4
0,0
6,0
9,5.19615
3,5.19615
6
0,0,3,0.16667
6,0,3,0.33333
9,5.19615,3,0.16667
3,5.19615,3,0.33333
3,1.73205,0.46410,1
6,3.46410,0.46410,1
11.00000
0.95031
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Ritangle 2025 Stage 3 Solutions

The winning submission arrived at 0709 on the final day of the competition. The winning team, 'Fudge Factors' from King's College London Mathematics School, devised a strategy that asymptotically approaches a P-value of around 115.37 and a Q-value of 0.9913.

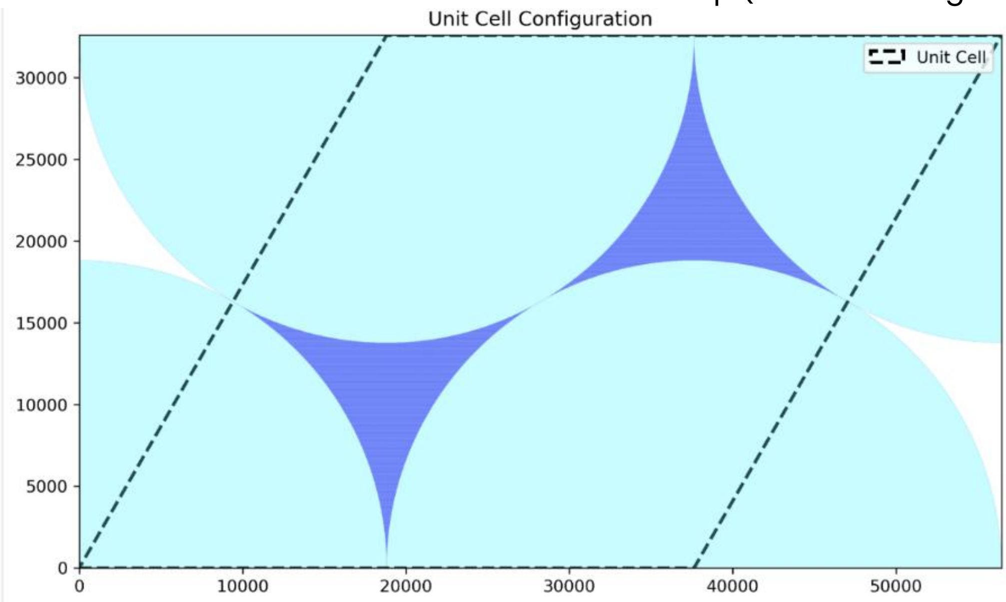
Their method utilised a fractal-like packing strategy using three disc radii $r_1 \gg r_2 \gg r_3$ where r_3 is very close to zero (dust). By first placing a hexagonal packing of r_1 discs (density ≈ 0.9069) and filling the gaps with a second hexagonal packing of radius r_2, they achieved a theoretical area coverage of $1-(1-0.9069)^2 \approx 0.9913$. The dust discs ($r_3$) then covered the remaining grid points. In the limit where the radii are sufficiently separated in magnitude, the disc count is dominated by these smallest discs, causing the P-value to approach $\frac{1}{1-0.9913} \approx 115.37$. To align this hexagonal geometry with the integer grid, they used a Pythagorean triple with an angle approximating 60° to define the cell vectors.

The solution they submitted had a P-value of 102.55961 and a Q-value of 0.99056 .

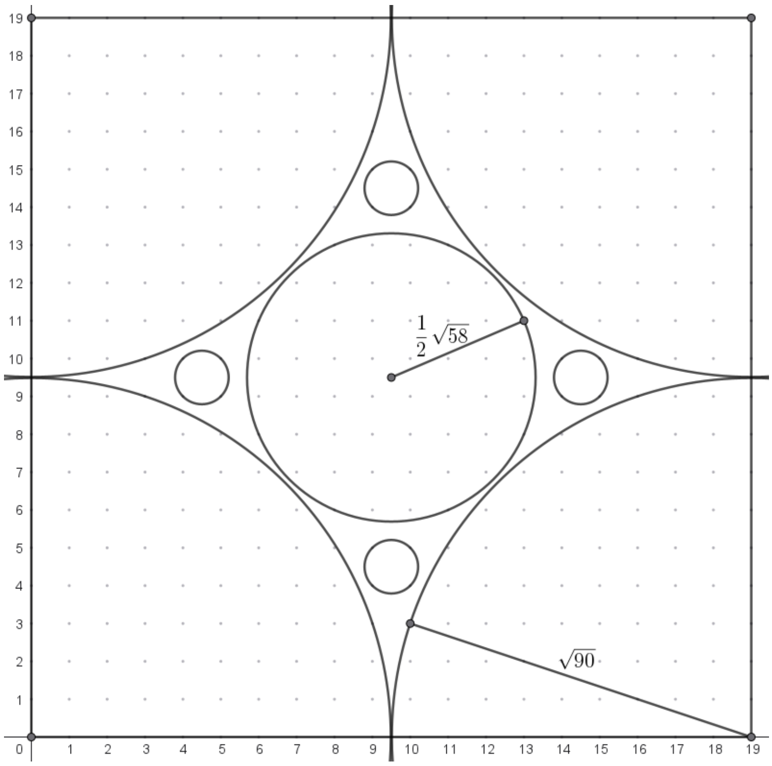
Their cell was a parallelogram with corners at $(0,0)$, $(37633,0)$, $(56448,32592)$ and $(18815,32592)$.

The cell included:

- 1 complete disc with radius 18816.50000,
- $386{,}022$ discs with radius 9.20000, and
- $11{,}573{,}215$ discs with radius 0.00000 when rounded to 5dp (each covering one grid point).



Until 24 hours before the end of the competition, the best answer received was as shown below; this has $P = 60.16667$ and $Q = 0.92681$. (Equivalent submissions centred their cell on one of this cell's vertices.)



This would be entered as
4
0,0
19,0
19,19
0,19
9
0,0,9.48683,0.25
19,0,9.48683,0.25
19,19,9.48683,0.25
0,19,9.48683,0.25
9.5,9.5,3.80789,1
9.5,4.5,0.70711,1
14.5,9.5,0.70711,1
9.5,14.5,0.70711,1
4.5,9.5,0.70711,1
60.16667
0.92681