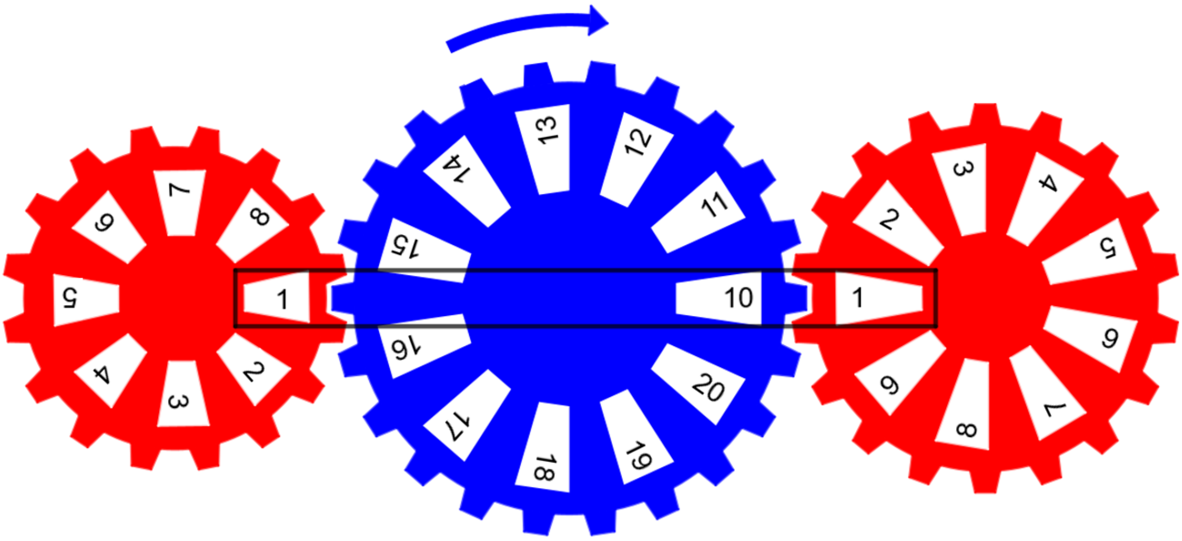
Ritangle 2025 Stage 1 and 2 Questions

At the close of the competition, 454 teams had unlocked Stage 3 using the 28 clues

Qn	Total number of attempts	Number of teams with correct answer
1	4366	1657
2	6470	1511
3	11807	1378
4	4962	1258
5	5000	1189
6	7530	1060
7	7374	974
8	4144	1063
9	5723	1010
10	2420	980
11	3884	905
12	2198	964
13	4774	931
14	2467	919
15	2066	909
16	4657	798
17	2779	851
18	2093	863
19	4353	782
20	5996	711

Qn	Total number of attempts	Number of teams with correct answer
21	2454	708
22	4197	616
23	5552	619
24	2524	632
25	5801	624
26	2867	659
27	4187	614
28	2193	657

Question 1



As the cogs rotate, with the central cog always going clockwise, Ada adds together all 792 different 4-digit numbers she sees in the rectangle, starting with 1101 as shown. Her sum is S.

Will, on the other hand, multiplies the three numbers in the rectangle together. The sum of these 792 products is T.

What is the value of S/T (to 4 s.f.)?

Solution

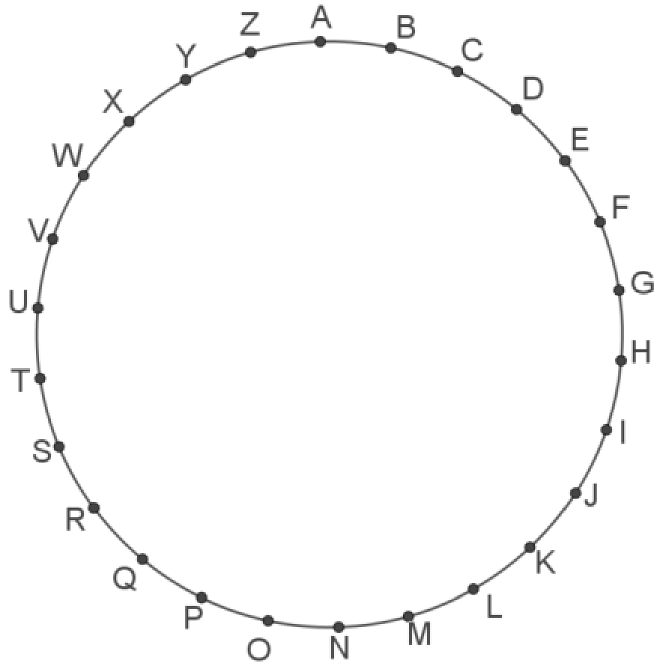
$$S = (1+2+3+4+5+6+7+8) \times 1000 \times 99 + (10+11+12+13+14+15+16+17+18+19+20) \times 10 \times 72 + (1+2+3+4+5+6+7+8+9) \times 88 = 3686760$$

$$T = (1+2+3+4+5+6+7+8) \times (10+11+12+13+14+15+16+17+18+19+20) \times (1+2+3+4+5+6+7+8+9) = 267300$$

Answer 13.79

Clue: 267

Question 2



The 26 letters of the alphabet are equally spaced, in alphabetical order, around this circle.

The shortest route along the circle from R to I is 9 steps (anticlockwise).

The shortest route from I to T is 11 steps (clockwise).

To spell RITANGLE in this way, starting at the first letter and always using the shortest route between each pair of successive letters, requires 9+11+7+13+7+5+7=59 steps. Call this the path length of RITANGLE.

The path length of INTEGRAL is 55 steps (and every step is clockwise!)

Consider the path length for each of the 8! arrangements of the letters R-I-T-A-N-G-L-E.

If the shortest path length is x steps and the longest path length is y steps, what is $x \times y$?

Solution

The shortest path AEGILNRT (or TRNLIGEA) has length 19 steps.
The longest path comes from aiming for diametrically opposite letters.
LANERGTI has 81 steps.
19*81 =1539
Answer 1539

Clue: 851

Question 3

You are going to roll five fair six-sided dice with sides labelled 1 to 6.

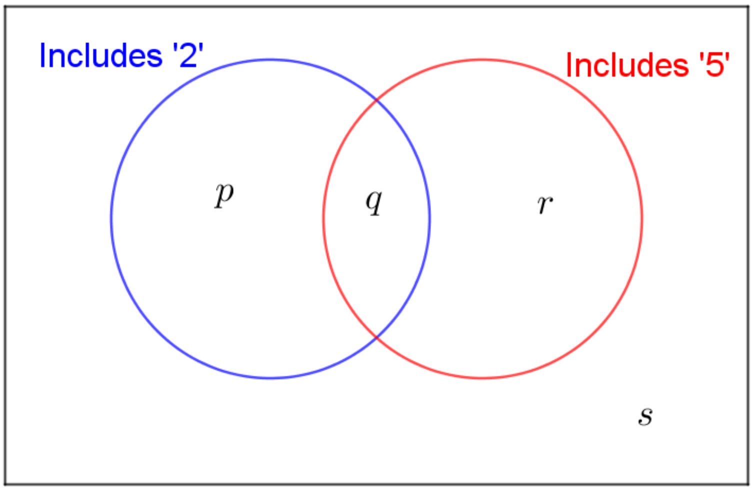
Your aim is to insert your five scores into the boxes below to create a 5-digit number (i.e. a number in base 10 between 11111 and 66666) that is divisible by 25.

□□□□□

What is the probability (to 3d.p.) that this is possible?

Solution

The only numbers divisible by 25 end 00,25,50,75.
Only '25' is possible with dice scores.



$$P(At\ least\ one\ 2) = 1 - P(no\ 2s) = 1 - \left(\frac{5}{6}\right)^5 = p + q$$

$$P(At\ least\ one\ 5) = 1 - P(no\ 5s) = 1 - \left(\frac{5}{6}\right)^5 = q + r$$

$$P(no\ 2s\ or\ 5s) = \left(\frac{4}{6}\right)^5 = s$$

$$q = (p+q) + (q+r) + s - 1 = 1 - \left(\frac{5}{6}\right)^5 + 1 - \left(\frac{5}{6}\right)^5 + \left(\frac{4}{6}\right)^5 - 1 = \frac{6^5 - 2 \times 5^5 + 4^5}{6^5} = \frac{2550}{6^5}$$

Answer 0.328

Clue: 259

Question 4

A 5 by 15 by 27 cuboid is made up from 2025 unmarked wooden unit cubes.

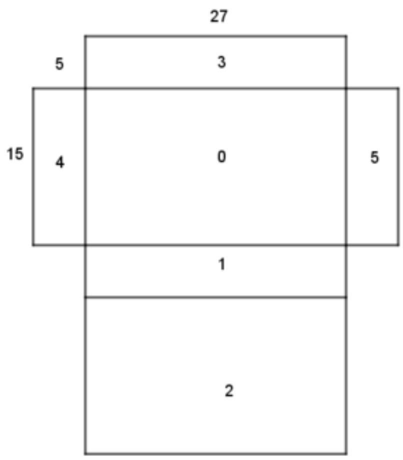
Each face of the cuboid is assigned a different integer from 0 to 5 and that number is written on every unit square making up that face.

The 2025 cubes are separated.

The value of a unit cube is the sum of the numbers (if any) on its faces.

Given that the sum of all the numbers on all the faces is 2025, how many cubes have an odd value?

Solution



Value		
1	1-0 1	25+75=100
3	3-0 2-1 3	25+25+75=125
5	4-1-0 4-1 3-2 5 5-0	1+3+25+39+13=81
7	4-2-1 4-3-0 5-2 4-3	1+1+13+3=18
9	4-3-2	1=1
		Total 325

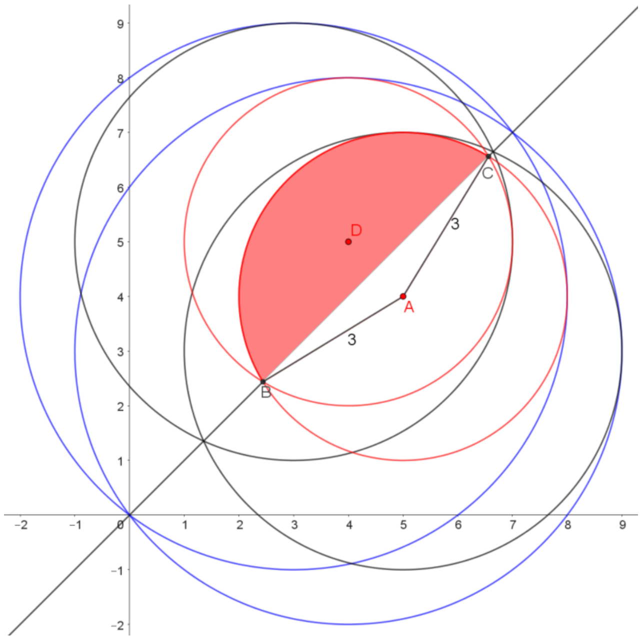
Answer 325

Clue: 433

Question 5

Find the area (to 3.d.p.) of the region made up of all the points that are inside all six circles of the form $(x-a)^2+(y-b)^2=c^2$ where $\{a,b,c\}=\{3,4,5\}$.

Solution



It can be seen that the required region is that inside the two circles of radius 3.

Since the distance between the centres, AD, is $\sqrt{2}$, the angle $CAB=\alpha=2\times\arccos\left(\frac{\sqrt{2}}{6}\right)$

The required area is $2\times\frac{9}{2}(\alpha-\sin\alpha)$

Answer 19.868

Clue: 493

Question 6

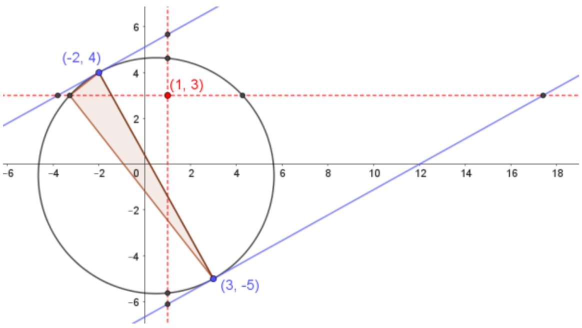
The triangle with vertices A(1,3), B(-2,4) and C(3,-5) is not right-angled.

Changing only one of these six coordinates can result in ABC being the vertices of a right-angled triangle.

Find (to 3d.p.) the area of the smallest right-angled triangle that can be created in this way.

Solution

The smallest area arises from moving A so that the right angle is at A $\left(\frac{1-\sqrt{57}}{2}, 3\right)$ as shown here. The area is 8.237 square units.



Answer 8.237

Clue: 165

Question 7

09:19:29 and 12:11:10 is each an example of what we will call an Arithmetic Time. These are times in the 24 hour clock, H:M:S, when the integers H,M,S, in that order, form an arithmetic sequence with a non-zero common difference.

On an analogue clock, let

$0° \leq \alpha < 180°$ be the angle between the hour and minute hands,

$0° \leq \beta < 180°$ be the angle between the minute and second hands, and

$0° \leq \gamma < 180°$ be the angle between the second and hour hands.

What is the minimum value of $\alpha + \beta + \gamma$ (in degrees to 1d.p.) for an Arithmetic Time?

Solution

At 15:16:17 the angle measuring clockwise from the 3:

For S: $\frac{2}{60} \times 360° = 12°$

For M: $\frac{1}{60}\left(1 + \frac{17}{60}\right) \times 360° = 7.7°$

For H: $\frac{1}{12}\left(0 + \frac{16}{60} + \frac{17}{60^2}\right) \times 360° = 8\frac{17}{120}°$

At this time $\alpha + \beta + \gamma = (12 - 7.7) + \left(12 - 8\frac{17}{120}\right) + \left(8\frac{17}{120} - 7.7\right) = 24 - 15.4 = 8.6°$

Answer 8.6

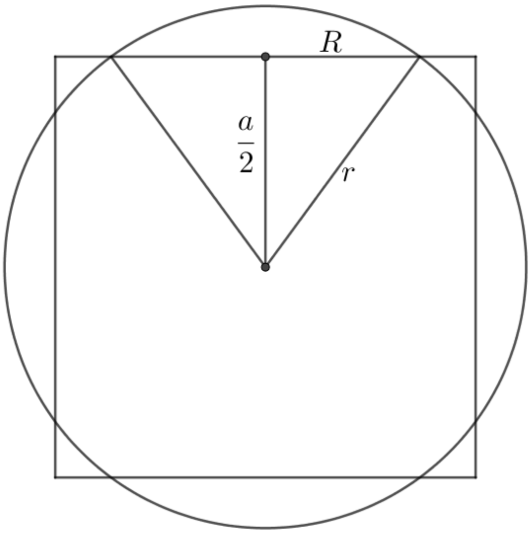
Clue: 701

Question 8

A cube and a sphere have the same volume and same centre. If p% of the surface area of the cube lies outside the sphere, what is p (to 3 sig.figs.)?

Solution

If the cube has side length a and the sphere radius r then $\frac{4}{3}\pi r^3 = a^3$



The radius, R, of the circle of intersection on the top face is given by $R^2 = r^2 - \left(\frac{a}{2}\right)^2$

The proportion required is $(a^2 - \pi R^2)/a^2 \quad = 1 - \frac{\pi\left(r^2 - \left(\frac{a}{2}\right)^2\right)}{a^2} = 1 - \frac{\pi}{a^2}\left(\left(\frac{3}{4\pi}\right)^{\frac{2}{3}} a^2 - \frac{a^2}{4}\right)$

$$= 1 - \pi\left(\left(\frac{3}{4\pi}\right)^{\frac{2}{3}} - \frac{1}{4}\right) = 0.576404$$

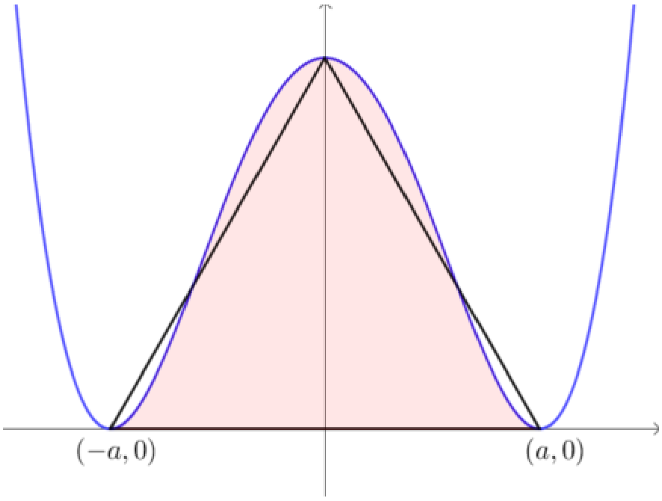
Answer 57.6

Clue: 102

Question 9

A quartic function has two minimum turning points on the x-axis as shown and the three turning points are at the corners of an equilateral triangle.

If the area of the region under the curve as shown is exactly 1 square unit more than the area of the equilateral triangle, what is the value of a to 3d.p.?



Solution

Quartic must have equation of the form $y = k(x-a)^2(x+a)^2$ and pass through $(0, a\sqrt{3})$

Therefore $ka^4 = a\sqrt{3}$ and so $k = \frac{\sqrt{3}}{a^3}$.

The equilateral triangle has area $\frac{1}{2}(2a)^2 \sin 60° = a^2\sqrt{3}$

From the given area property: $\int_{-a}^{a} k(x-a)^2(x+a)^2 dx = 1 + a^2\sqrt{3}$

$$2k\left[\frac{x^5}{5} - 2a^2\frac{x^3}{3} + a^4x\right]_0^a = 1 + a^2\sqrt{3}$$

$$\frac{16}{15}\frac{\sqrt{3}}{a^3}a^5 = 1 + a^2\sqrt{3}$$

$$a = \sqrt{5\sqrt{3}}$$

Note, this makes the area under the curve 16 and the equilateral triangle has area 15.

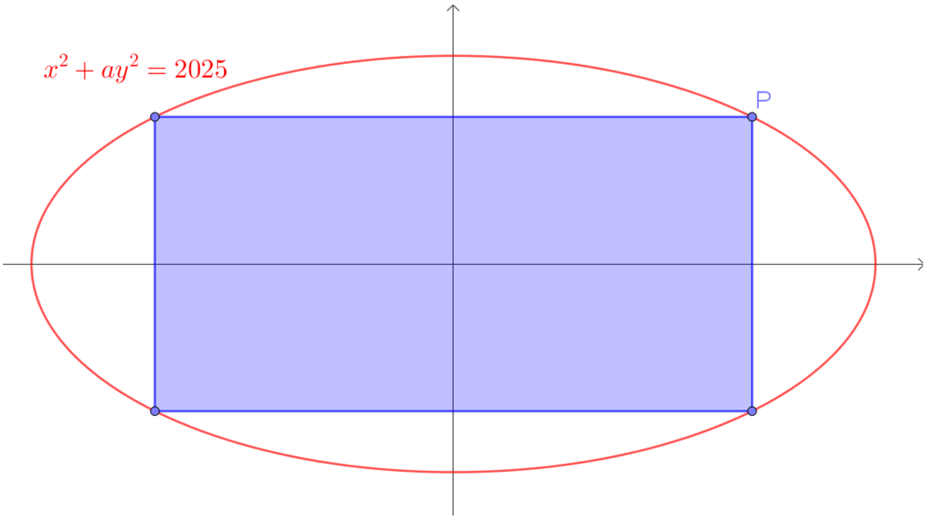
Answer 2.943

Clue: 914

Question 10

The point P lies on the ellipse with equation $x^2 + ay^2 = 2025$.

P is reflected in the x-axis to give point P′, and both P and P′ are reflected in the y-axis. These four points are corners of a rectangle as shown.



If the maximum possible value of the area of a rectangle formed in this way is 2000 square units, what is the value of a (to 4 d.p.)?

Solution

For P: (x,y) the area is given by A=4xy so $A^2 = 16y^2(2025 - ay^2)$

The maximum value of A^2 occurs at the same point as the maximum of A and this is when $2025 \times 32y - 64ay^3 = 0$

I.e $2025 - 2ay^2 = 0$ and so $\frac{2025}{2} = ay^2$ and this gives $\frac{2025}{2} = x^2$

Therefore the maximum area is $A = 4 \times \sqrt{\frac{2025}{2}} \times \sqrt{\frac{2025}{2a}} = 2000$

$A = 4 \times \frac{2025}{2\sqrt{a}} = 2000$ so $\sqrt{a} = \frac{4050}{2000}$ and $a = \left(\frac{81}{40}\right)^2 = 4.1006$

Alternative approach would be to use parametric equations:

$P: \left(45cost, \frac{45}{\sqrt{a}}sint\right)$ and so we want to maximise $A = 4 \times 45cost \times \frac{45}{\sqrt{a}}sint = \frac{4050}{\sqrt{a}}sin2t$. This occurs when $sin2t = 1$ and so $\frac{4050}{\sqrt{a}} = 2000$.

Answer 4.1006

Clue: 494

Question 11

The line segment with end points (1,2) and (12,*a*) (where *a*>2) is rotated through 360° about the x-axis to create a frustum of a cone.

The same line segment is rotated through 360° about the y-axis to create another frustum of a cone.

If the difference in volumes of these two frustums is 50 cubic units, find the sum (to 3d.p.) of the possible values of *a*.

Solution

$$\frac{\pi}{3}(a-2)(12^2+1^2+12)-\frac{\pi}{3}\times 11\times(a^2+2^2+2a)=\pm 50$$

$$11a^2-135a+358+\frac{150}{\pi}=0 \quad \text{and} \quad 11a^2-135a+358-\frac{150}{\pi}=0$$

The sum of the roots of these quadratics in both cases is 135/11 (you can solve to see all roots satisfy *a*>2) so the required sum is 270/11=24.545

Answer 24.545

Clue: 468

Question 12

The parameters a, b and c each take the value 0 or 1.

There are eight such triples to consider.

Find (to 3 d.p.) the value of the expression

$$\sum_{a,b,c} \int_{-1}^{c} (x-a)^b (x+b)^c x^a \, \mathrm{d}x$$

Solution

a	b	C		
0	0	0	$\int_{-1}^{0} 1 \, \mathrm{d}x$	1
0	0	1	$\int_{-1}^{1} x \, \mathrm{d}x$	0
0	1	0	$\int_{-1}^{0} x \, \mathrm{d}x$	-1/2
0	1	1	$\int_{-1}^{1} x(x+1) \, \mathrm{d}x$	2/3
1	0	0	$\int_{-1}^{0} x \, \mathrm{d}x$	-1/2
1	0	1	$\int_{-1}^{1} x^2 \, \mathrm{d}x$	2/3
1	1	0	$\int_{-1}^{0} (x-1)x \, \mathrm{d}x$	1/3+1/2
1	1	1	$\int_{-1}^{1} (x-1)(x+1)x \, \mathrm{d}x$	0
			Total	13/6

Answer 2.167

Clue: 460

Question 13

I have a stopwatch that records time in the form HH:MM:SS.

It resets to 00:00:00 after 100 hours.

During this 100 hour interval, how many times will the display use precisely two different digits (like 00:00:01 and 99:59:59)?

Solution

Both digits from {0,1,2,3,4,5} $\left(2^6-2\right)\times{}^6C_2$=62*15=930

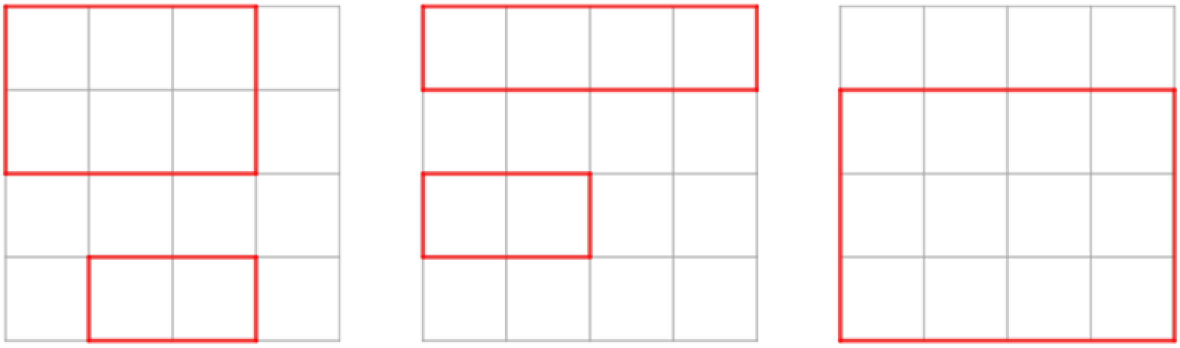
One of digits from 6,7,8,9: $\left(2^4-1\right)\times 6\times 4$=360

Answer 1290

Clue: 143

Question 14

A 4 by 4 square grid contains 16 unit squares and, with gridlines for edges, a total of 35 landscape rectangles (i.e. a rectangle whose width is greater than its height). Five of these 35 rectangles are shown below.



A 45 by 45 square grid contains 2025 unit squares.

With gridlines for edges, how many landscape rectangles does it contain?

Solution

$\binom{46}{2}^2$ rectangles in total since each rectangle is uniquely determined by its two vertical edges and two horizontal edges and there are $\binom{46}{2}$ pairs of each.

The number of squares is $\sum_1^{45} r^2 = \frac{45\times46\times91}{6}$

So the number of oblongs is $\left(\frac{46\times45}{2}\right)^2 - \frac{45\times46\times91}{6}$

We need half of these.

Answer 519915

Clue: 150

Question 15

The curve $log_2 x = log_{\,3} y$ passes through the points $(1,1)$ and $(8,27)$

Let A be the area of the region between the curve, the x-axis and the lines $x = 1$ and $x = 8$

Let B be the area of the region between the curve, the y-axis and the lines $y = 1$ and y$= 27$

What is the value of $|A - B|$ to 3d.p.?

Solution

Method 1

$log_2 x = log_{\,3} y$ can be written $\frac{lnx}{ln2} = \frac{lny}{ln3}$ which rearranges to $lny = lnx\frac{ln3}{ln2}$ and so $y = x^{\left(\frac{ln3}{ln2}\right)}$

Then $\int_1^8 ydx = \left[\frac{x^{\left(\frac{ln3}{ln2}\right)+1}}{\left(\frac{ln3}{ln2}\right)+1}\right]_1^8 = \left[\frac{ln2 \times x^{\left(\frac{ln6}{ln2}\right)}}{ln6}\right]_1^8 = \frac{ln2}{ln6}\left(8^{\left(\frac{ln6}{ln2}\right)} - 1\right) = 83.17335$

And $\int_1^{27} xdy = \left[\frac{y^{\left(\frac{ln2}{ln3}\right)+1}}{\left(\frac{ln2}{ln3}\right)+1}\right]_1^{27} = \left[\frac{ln3 \times y^{\left(\frac{ln6}{ln3}\right)}}{ln6}\right]_1^{27} = \frac{ln3}{ln6}\left(27^{\left(\frac{ln6}{ln3}\right)} - 1\right) = 131.8266$

Answer 48.653

Method 2

$log_2 x = log_{\,3} y = t \Rightarrow x = 2^t, y = 3^t$

$$A = \int_0^3 y\frac{dx}{dt}dt = \int_0^3 3^t 2^t \ln 2\, dt = ln2\int_0^3 6^t dt = ln2\int_0^3 e^{tln6}dt = \frac{ln2}{ln6}[e^{tln6}]_0^3 = 215\frac{ln2}{ln6}$$

Similarly $B = \int_0^3 x\frac{dy}{dt}dt = 215\frac{ln3}{ln6}$

$$|A - B| = 215\frac{ln1.5}{ln6}$$

Answer 48.653

Clue: 832

Question 16

Two convergent geometric sequences, $\{u_n\}_{n\geq 1}$ and $\{v_n\}_{n\geq 1}$, of positive terms (common ratio, $0 < r < 1$) are determined by the following values:

$$u_1 = 6, u_3 = 4 \quad \text{and} \quad v_1 = 7, v_2 = 5$$

The difference, D, between their sums to infinity is $\left| \frac{6}{1-\sqrt{\frac{2}{3}}} - \frac{7}{1-\frac{5}{7}} \right| = 8.197$ (to 4 sig.figs)

Place the integers 2 to 7 in the boxes below, using each integer once, to determine two convergent geometric sequences of positive terms for which the value of D is as large as possible.

$$u_1 = \square, u_{\square} = \square \qquad v_1 = \square, v_{\square} = \square$$

What is this maximum value of D to 4 sig.figs?

Solution

Max is D=186.812 for $u_1 = 6, u_7 = 5$ $\qquad v_1 = 4, v_3 = 2$

Min is D=0.35089 for $u_1 = 5, u_4 = 3$ $\qquad v_1 = 7, v_6 = 2$

Answer 186.8

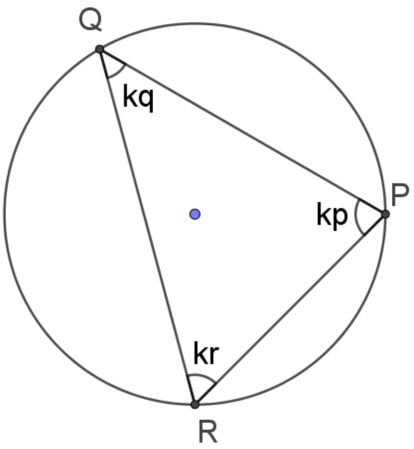
Clue: 580

Question 17

I roll three fair, six-sided dice with sides labelled 1 to 6 and the scores are p, q and r (which are not necessarily distinct).

I construct a triangle PQR with vertices on a circle, and with interior angles in the ratio p:q:r.

What is the probability (to 3.d.p.) that the centre of the circle lies neither inside nor on an edge of the triangle?



Solution

The triangle must be obtuse angled (think circle theorems) and so the highest score must be greater than the sum of the other two.

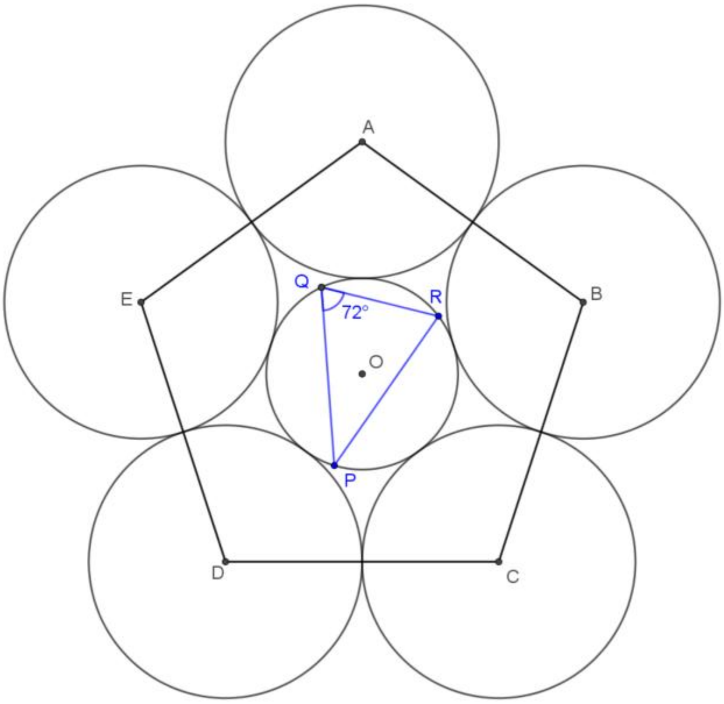
Max score		**Number of permutations**
3	(3,1,1)	3
4	(4,1,1), (4,1,2)	3+6
5	(5,1,1), (5,1,2), (5,1,3), (5,2,2)	3+6+6+3
6	(6,1,1), (6,1,2), (6,1,3), (6,1,4), (6,2,2),(6,2,3)	3+6+6+6+3+6

Therefore the required probability is 60/216=5/18

Answer 0.278

Clue: 299

Question 18



Circles with unit radius are centred on the vertices of a regular pentagon, ABCDE, of side length 2 units.

A sixth circle, centre O, touches each of these circles. Triangle PQR has its vertices on this circle. If angle PQR is 72°, what is the length (to 3d.p.) of side PR?

Solution

From isosceles triangle DOC we have $DO = \frac{1}{\cos 54°}$
and so the central circle has diameter $2\left(\frac{1}{\cos 54°} - 1\right)$

Move Q on the circle to make PQ a diameter. By the circle theorems the angle at Q does not change and the angle at R is a right angle.

Therefore $\sin 72° = \frac{PR}{PQ}$ and so PR=$2\sin 72°\left(\frac{1}{\cos 54°} - 1\right)$

Answer 1.334

Clue: 334

Question 19

Primrose chooses a prime number between 2 and 98 inclusive at random and calls it P.

Eve chooses an even number between 2 and 98 inclusive at random and calls it E.

Simultaneously they recite successive terms in their own arithmetic sequence:

Primrose has first term P and common difference E.

Eve has first term 1000E and common difference -P.

What is the probability (to 3 sig.figs) that, at some stage, they say the same number at the same time?

Solution

$$P + nE = 1000E - nP \iff n(E+P) = 1000(E+P) - 1001P \iff n = 1000 - \frac{1001P}{(E+P)}$$

Therefore, for n to be an integer, E + P must be a factor of 1001P = 7*11*13*P

P	Factors of 1001P (greater than P +1 and less than P+99)	Number of values of E
2	7,11,13,77,91 (143 too big) and 14,22,26	3 (since E is even)
3	7,11,13,77,91 and 21,33,39	8
5	7,11,13,77,91 and 35,55,65	8
7	11,13,77,91 and 49	5
11	13,77,91	3
13-43	77,91	2 each: 18
47-73	77,91,143	3 each: 21
79-89	91, 143	2 each: 6
97	143	1

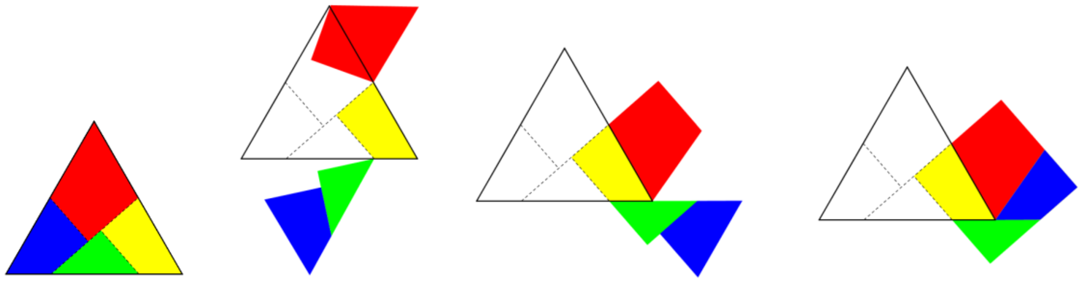
The number of ordered pairs (P,E) is therefore 73 and so probability is 73/(25*49)

Answer 0.0596

Clue: 408

Question 20

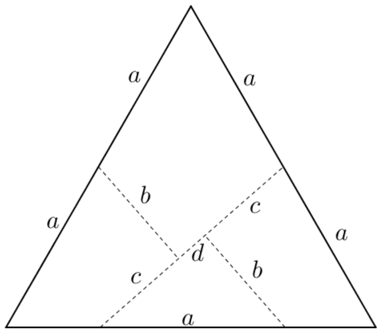
These images show an equilateral triangle that has been cut into four pieces and these pieces rotating about their corners to form a square.

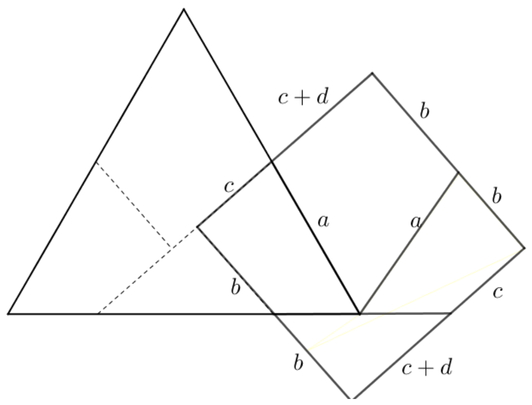


What is the size of the smallest angle (in degrees, to 4 sig. figs.) in the triangular piece?

Solution

Think about the edges that will match.





Areas: $\frac{1}{2}(2a)^2\frac{\sqrt{3}}{2}=(2b)^2 \quad \Rightarrow \quad a^2\sqrt{3}=4b^2 \quad \Rightarrow \quad \frac{\sqrt{3}}{4}=\left(\frac{b}{a}\right)^2$

$$\sin^{-1}\frac{b}{a}=\sin^{-1}\sqrt{\frac{\sqrt{3}}{4}}=41.2°$$

Answer 41.15

Clue: 296

Question 21

Let S be the set of positive integers that can be written in the form $2^a 3^b 5^c 7^d$, where a, b, c, d are non-negative integers. For example, all the integers from 1 to 10 are in S. What (to 4s.f.) is the sum of the reciprocals of all the numbers in S?

Solution

These numbers are all included once in the expansion

$$\left(1+\frac{1}{2}+\frac{1}{2^2}+\ldots\right)\left(1+\frac{1}{3}+\frac{1}{3^2}+\ldots\right)\left(1+\frac{1}{5}+\frac{1}{5^2}+\ldots\right)\left(1+\frac{1}{7}+\frac{1}{7^2}+\ldots\right)$$

Evaluating the infinite geometric series in each bracket gives

$$\frac{1}{1-\frac{1}{2}}\times\frac{1}{1-\frac{1}{3}}\times\frac{1}{1-\frac{1}{5}}\times\frac{1}{1-\frac{1}{7}}=2\times\frac{3}{2}\times\frac{5}{4}\times\frac{7}{6}=\frac{35}{8}=4.375$$

Answer 4.375

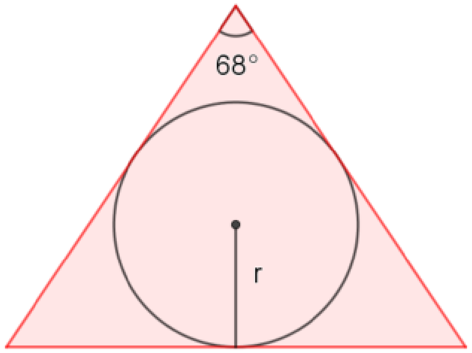
Clue: 763

Question 22

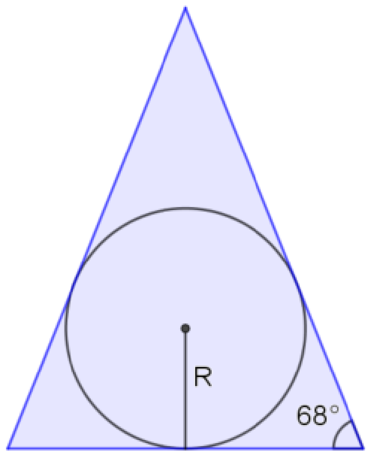
For any value of A with 60<A<90 there are two isosceles triangles with area A square units and **largest** interior angle A degrees.

The diagram shows the two isosceles triangles in the case A=68, along with their incircles.

For what value of A (where 60<A<90) is the sum of the areas of the two incircles the greatest? Give your answer to 2 d.p.

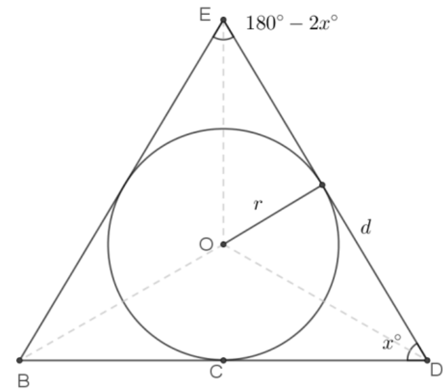


Area 68



Area 68

Solution



Let Δ represent the area of triangle BDE. $EB = ED = d$. Express Δ in two ways:

$$\Delta = \frac{1}{2}d^2 sin(180° - 2x°) = \frac{1}{2}d^2 sin(2x°) \qquad (1)$$

$$\Delta = \frac{1}{2}BD \times r + \frac{1}{2}DE \times r + \frac{1}{2}EB \times r = \frac{r}{2}(BD + DE + EB) = \frac{r}{2}(2d\cos x° + 2d) \qquad (2)$$

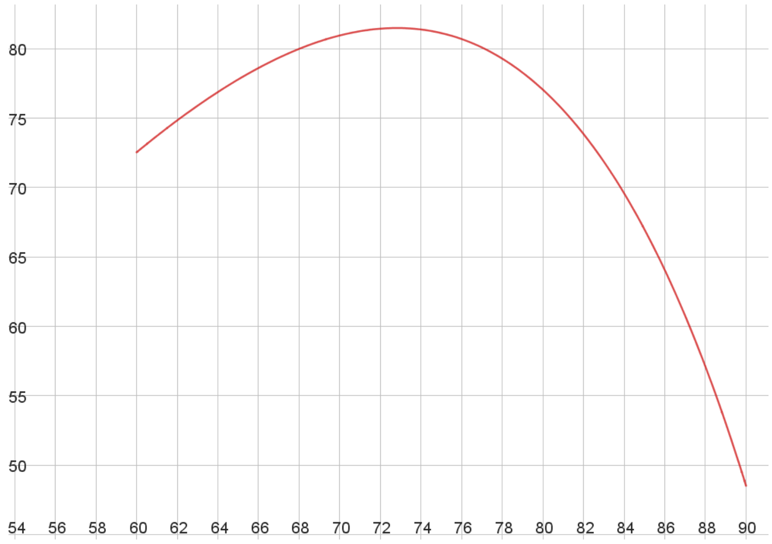
From (2), $\Delta = rd(1 + \cos x°)$ and substituting for d in (1) gives $\Delta = \frac{\Delta^2 sin(2x°)}{2r^2(1+\cos x°)^2}$

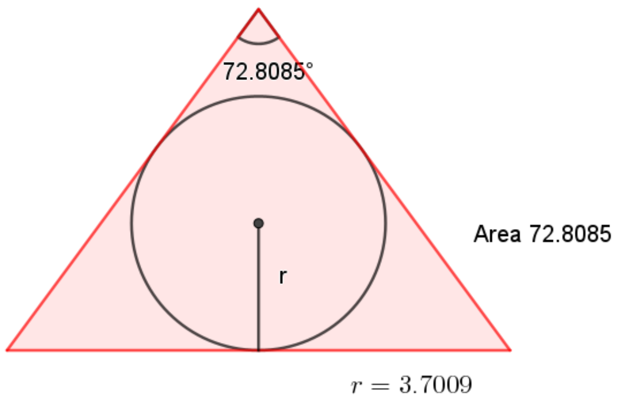
Rearranging, the area of the circle is $\pi r^2 = \pi\frac{\Delta\sin(2x°)}{2(1+\cos x°)^2}$

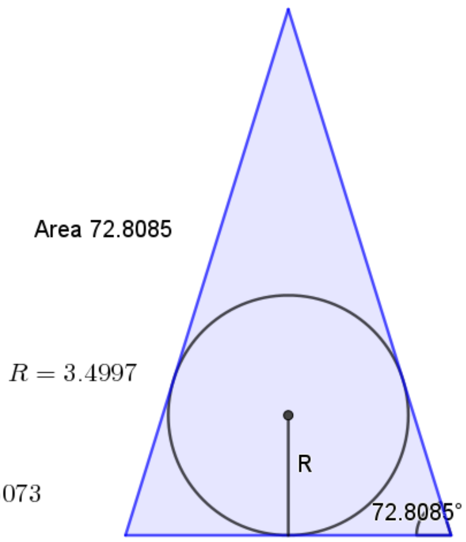
Therefore, the total area of the two circles in terms of A (recall $A = \Delta$):

$$\frac{\pi A}{2}\left(\frac{\sin 2\left(90^\circ-\frac{A^\circ}{2}\right)}{\left(1+\cos\left(90^\circ-\frac{A^\circ}{2}\right)\right)^2}+\frac{\sin(2A^\circ)}{(1+\cos(A^\circ))^2}\right)=\frac{\pi A}{2}\left(\frac{\sin(A^\circ)}{\left(1+\sin\left(\frac{A^\circ}{2}\right)\right)^2}+\frac{\sin(2A^\circ)}{(1+\cos(A^\circ))^2}\right)$$

Graphing software gives the optimal value of A when A=72.81







Sum of areas of circles $= 43.0288 + 38.4784 = 81.5073$

Answer 72.81

Clue: 155

Question 23

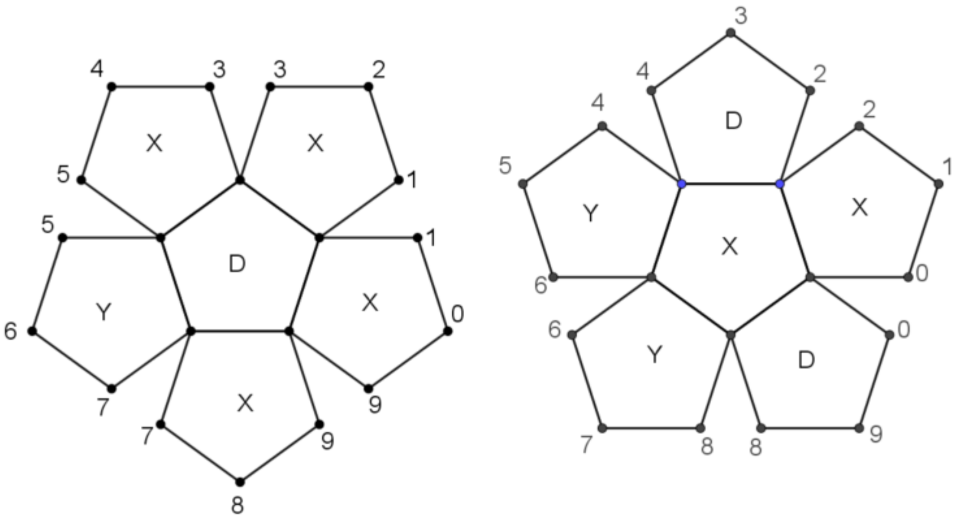
The 12 letters D,O,D,E,C,A,H,E,D,R,O,N are randomly assigned to the 12 faces of a regular dodecahedron, one letter per face.

What is the probability (to 3.d.p.) that no two faces labelled D share an edge but the two faces labelled O do share an edge?

Solution

After assigning the first D, place that face on the bottom. The top face cannot be assigned D as that would make the third D impossible to place. So there are 5 choices (the 'top ring') for the second D and that leaves 2 choices for the third D.

$$P(No\ 2\ Ds\ share\ an\ edge) = \frac{5}{11} \times \frac{2}{10}$$



There is essentially only one way to place the three Ds, as show above.

Any three faces meet at one vertex; 10 of the vertices are labelled above.

If the first O occupies a face labelled X then there are 3 choices of face for the second O to occupy an adjoining face.

If the first O occupies a face labelled Y then there are 4 choices of face for the second O to occupy an adjoining face.

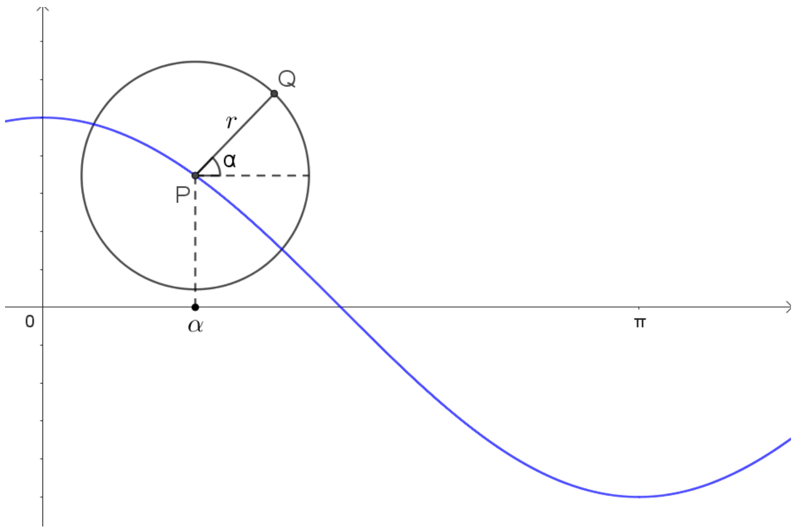
So the required probability is $\frac{5}{11} \times \frac{2}{10} \times \left(\frac{6}{9} \times \frac{3}{8} + \frac{3}{9} \times \frac{4}{8}\right) = \frac{5}{132}$

Answer 0.038

Clue: 145

BM 11/12/25
Version 1.0

Question 24

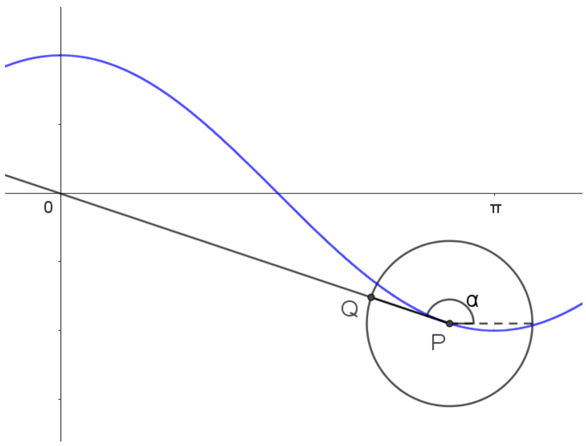


A point P moves along the curve $y = cosx$. P is the centre of a circle of radius r (r>0).

When P has coordinates $(\propto, cos \propto)$, the vector $\overrightarrow{PQ} = \begin{pmatrix} rcos\alpha \\ rsin\alpha \end{pmatrix}$ determines the position of point Q on the circle.

Given that Q passes through $(0,0)$ for a value of α where $0 < \alpha < 2\pi$, find the value of r (to 3d.p.)

Solution



α satisfies the equation $\tan(\pi - \alpha) = \frac{-cos\alpha}{\alpha}$

which is equivalent to $\frac{\sin\alpha}{\cos\alpha} = \frac{\cos\alpha}{\alpha}$ (gradient PQ =gradient OP).

Using numerical approximation or graphing software, $\alpha = 2.81704$

Then $r = \sqrt{\alpha^2 + \cos^2(\alpha)} = 2.97221$

Answer 2.972

Clue: 168

Question 25

$a(= 10a_1 + a_2),\ b(= 10b_1 + b_2),\ c(= 10c_1 + c_2)$ and $d(= 10d_1 + d_2)$ are different integers between 01 and 99 inclusive, with $a > b$.

$0 \leq a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2 \leq 9$

Enter the 8-digit string $a_1a_2b_1b_2c_1c_2d_1d_2$ which produces the maximum possible integer value of $(\sqrt{a} + \sqrt{b})(\sqrt{c} - \sqrt{d})$

Solution

$(\sqrt{72} + \sqrt{50})(\sqrt{98} - \sqrt{02}) = (6\sqrt{2} + 5\sqrt{2})(7\sqrt{2} - \sqrt{2}) = 11\sqrt{2} \times 6\sqrt{2} = 132$

Answer 72509802

Clue: 437

Question 26

The integers 1 to 10 are shared between two people. They each work out the product of their 5 integers. The highest common factor of the two products is h. What is the sum of all the different possible values of h?

Solution

The prime factorisation of $10! = 2^8 3^4 5^2 7$ so any possible HCF is of the form $2^a 3^b 5^c$ where $0 \leq a \leq 4, 0 \leq b \leq 2, 0 \leq c \leq 1$.

hcf	example
1	impossible
2^1	(3,6,9,1,7)
2^2	(2,5,10,1,7)
2^3	(4,5,10,1,7)
2^4	(8,5,10,1,7)
3^1	Impossible
$2^1 3^1$	(3,5,10,1,7)
$2^2 3^1$	(6,5,10,1,7)
$2^3 3^1$	(2,4,3,1,7)
$2^4 3^1$	(2,4,6,1,7)
3^2	Impossible
$2^1 3^2$	(10,5,9,1,7)
$2^2 3^2$	(2,6,3,1,7)
$2^3 3^2$	(4,2,9,1,7)
$2^4 3^2$	(8,2,9,1,7)

hcf	example
5^1	impossible
$2^1 5^1$	(3,6,9,5,1)
$2^2 5^1$	(3,6,9,10,1)
$2^3 5^1$	(4,2,5,1,7)
$2^4 5^1$	(8,2,5,1,7)
$3^1 5^1$	(2,4,6,8,10)
$2^1 3^1 5^1$	(2,3,5,1,7)
$2^2 3^1 5^1$	(4,3,5,1,7)
$2^3 3^1 5^1$	(8,3,5,1,7)
$2^4 3^1 5^1$	(8,6,5,1,7)
$3^2 5^1$	impossible
$2^1 3^2 5^1$	(2,9,5,1,7)
$2^2 3^2 5^1$	(4,9,5,1,7)
$2^3 3^2 5^1$	(8,9,5,1,7)
$2^4 3^2 5^1$	(8,9,5,2,7)

Answer $2355 = (1 + 2 + 2^2 + 2^3 + 2^4)(1 + 3 + 3^2)(1 + 5) - (1 + 3 + 9 + 5 + 45)$

Clue: 733

Question 27

A suspended sphere of ice is melting in such a way that its radius is decreasing constantly by 1mm per day, whilst retaining its spherical shape. The water drips into a container that is in the form of a regular tetrahedron with a vertex on the ground and its upper open face parallel to the ground. The sphere initially has a radius of 1 metre and the tetrahedral container is empty. At the moment when the radius is 99cm what is the rate of change (in mm/hour to 4.sig. figs) of the height of water in the container?

Solution

For the sphere: $V_S = \frac{4}{3}\pi r^3$ so $\frac{dV_S}{dr} = 4\pi r^2$ and we're given $\frac{dr}{dt} = -0.1$ (cm/day)

Therefore $\frac{dV_S}{dt} = -0.4\pi r^2$ and when $r = 99$

The water already lost has volume $\frac{4}{3}\pi(100^3 - 99^3)$

A regular tetrahedron with edge length a, has base area $a^2\frac{\sqrt{3}}{4}$ and height $h = \frac{a\sqrt{6}}{3}$

So the volume would be $\frac{1}{3}a^2\frac{\sqrt{3}}{4} \times \frac{a\sqrt{6}}{3} = a^3\frac{\sqrt{2}}{12}$ which, since $h = \frac{a\sqrt{6}}{3}$, is $V_T = \frac{\sqrt{3}}{8}h^3$

Therefore $\frac{dV_T}{dh} = \frac{3\sqrt{3}}{8}h^2$ and we know $\frac{dV_T}{dt} = 0.4\pi \times 99^2$

The volume in the container is $\frac{4}{3}\pi(100^3 - 99^3)$ and so this is equal to $\frac{\sqrt{3}}{8}h^3$

Meaning $h^3 = \frac{32}{3\sqrt{3}}\pi(100^3 - 99^3)$

Therefore the required value, from $\frac{dV_T}{dt} = \frac{dV_T}{dh}\frac{dh}{dt}$ is

$\frac{dh}{dt} = \frac{0.4\pi \times 99^2}{\frac{3\sqrt{3}}{8}h^2}$ when $h^3 = \frac{32}{3\sqrt{3}}\pi(100^3 - 99^3)$

This gives 2.743441 cm/day or 1.1431 mm/hour

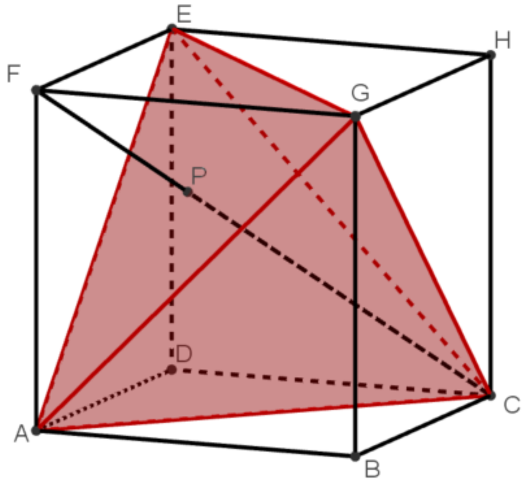
Answer 1.143

Clue: 154

Question 28

A unit cube and a regular tetrahedron share exactly three vertices. What (to 3d.p.) is the volume occupied by the tetrahedron that is outside the cube?

Solution



The pyramid ABCG has volume $\frac{1}{6}$. Therefore the tetrahedron has volume $1 - 4 \times \frac{1}{6} = \frac{1}{3}$

The tetrahedron we require is congruent to ACEG but with corner C reflected in face AEG. Therefore the volume required is $\frac{1}{3} - \frac{1}{6} = \frac{1}{6}$

Alternatively

The tetrahedron ACEG has side length, $AC = \sqrt{2}$, area of face AEG $\frac{\sqrt{3}}{2}$ and height PC $\frac{2\sqrt{3}}{3}$

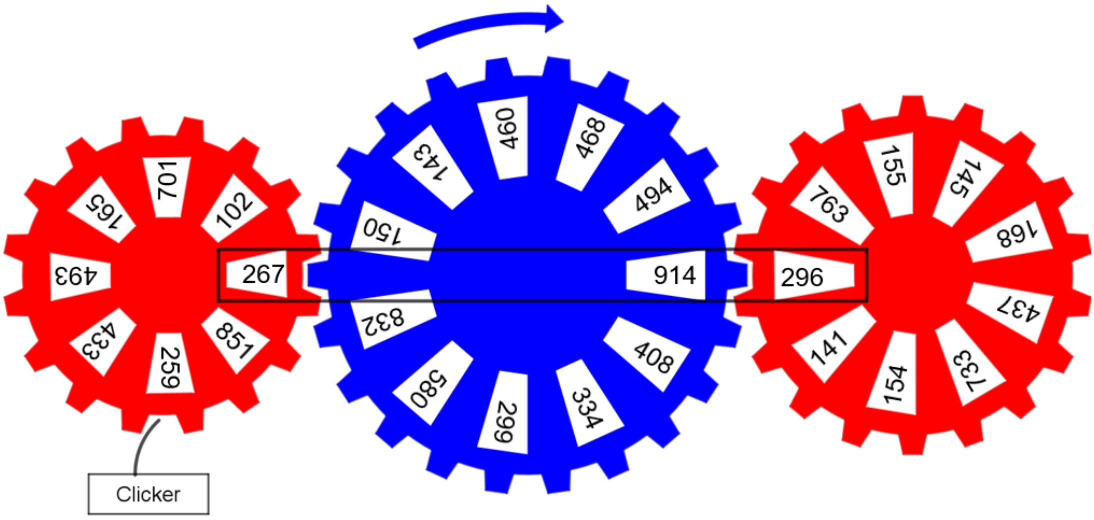
The diagonal, CF is $\sqrt{3}$ and so $PF = \sqrt{3} - \frac{2\sqrt{3}}{3} = \frac{\sqrt{3}}{3}$

The volume required is $\frac{1}{3}\frac{\sqrt{3}}{2}\left(\frac{2\sqrt{3}}{3} - \frac{\sqrt{3}}{3}\right) = \frac{1}{3}\frac{\sqrt{3}}{2}\left(\frac{\sqrt{3}}{3}\right) = \frac{1}{6}$

Answer 0.167

Clue: 141

Stage 3 unlock



Fibonacci numbers:	701408733, 267914296, 102334155, 433494437, 165580141
Pythagorean triples	493468155, 433408145
Triangles with 60° angle	259299155, 165832763
Triangles with integer area	851460437, 102150168, 165143154 plus the Pythagorean triples

Start with 701 - 408 - 733 in the rectangle.

Note that it takes 2 clicks to bring up each new triple.

493 - 468 - 155 after 28 clicks

259 - 299 - 155 after 540 more clicks

851 - 460 - 437 after 78 more clicks

102 - 334 -155 after 12 more clicks.

$28 \times 540 \times 78 \times 12 = 14152320$