FURTHER PURE MATHEMATICS WITH TECHNOLOGY: A POST-16 UNIT OF STUDY THAT USES TECHNOLOGY IN THE TEACHING, LEARNING AND ASSESSMENT

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Further Pure Mathematics with Technology is a new optional A level Mathematics unit that can be taken by pre-university students in England. The unit has been developed by Mathematics in Education and Industry, a mathematics education charity. It requires students to have access to technology, in the form of a graph-plotter, spreadsheet, programming language and computer algebra system (CAS) for the teaching, learning and assessment. This paper describes the development of the unit, including the rationale for the design decisions, and the implications for future developments of this type.

BACKGROUND

This first section is a background to Mathematics in Education and Industry (MEI) and gives some information about post-16 study in Mathematics in England so as to establish the context in which the Further Pure with Technology (FPT) unit was developed.

About MEI

Mathematics in Education and Industry (MEI) is a UK-based mathematics education charity whose primary focus is curriculum development. MEI has been innovating in mathematics education for nearly 50 years, with landmark projects such as the creation of the Further Mathematics Support Programme, which has been a major factor in the large increase in participation in post-16 mathematics in England (Stripp, 2010) and substantial continuing development programmes such as the Teaching Advanced Mathematics CPD course (MEI, 2005).

MEI seeks to ensure that the teaching and learning of mathematics in schools and colleges continues to reflect techniques that are relevant and practised in industry and across Higher Education. One area that continues to develop at a fast pace is that of technology. MEI is keen to understand how software and hardware may be used to motivate learning in mathematics.

Post-16 study of Mathematics in England: A level Mathematics and Further Mathematics

Education in England is currently compulsory up until the age of 16. Approximately half of all students continue to post-16 study of A levels and they will typically study three or four A levels over two years with the vast majority of these using this as preparation for university. There are no compulsory subjects at A level; however, about a quarter of students choose to study mathematics and of these students 16% also take a second, more demanding, A level in mathematics: A level Further Mathematics. In 2012 there were 78 951 students who took A level Mathematics and 12 688 who took A level Further Mathematics (JCQ, 2012a).

The study of A level Mathematics comprises of six units: four compulsory pure mathematics units plus two optional units that can be chosen from mechanics, statistics or discrete/decision mathematics. A level Further Mathematics also comprises of six units: two compulsory pure mathematics units plus four optional units that can be chosen from further pure mathematics, mechanics, statistics or discrete/decision mathematics. MEI has its own A level examination that is administered and assessed by OCR (Oxford, Cambridge and RSA examinations). Further Pure

Mathematics with Technology (FPT) has been developed as an optional unit that students can take as part of A level Further Mathematics.

DEVELOPING THE UNIT

Process of development

Although there have been a number of projects in England in the past 20-30 years to integrate the use of technology into the teaching and learning in mathematics it is not used effectively in many classrooms. This issue was raised by Ofsted (2008, p27):

> Several years ago, inspection evidence showed that most pupils had some opportunities to use ICT as a tool to solve or explore mathematical problems. This is no longer the case … despite technological advances, the potential of ICT to enhance the learning of mathematics is too rarely realised.

In the GCSE examinations that students take at age 16 they are allowed a scientific calculator, but not a graphical calculator, for some of the assessment. At A level students are allowed a graphical calculator in all but one of their examinations; however, these examinations are designed to be graphical calculator neutral, i.e. having a graphical calculator should offer no advantage to a student. It is not surprising that if the technology is expected to not offer an advantage in the examination then many teachers do not exploit its use for teaching and learning.

In addition to this there are no examinations where computer algebra systems (CAS) are allowed. The Joint Council for Qualifications requirements for conducting examinations (JCQ, 2012b) state:

> Calculators must not … be designed or adapted to offer any of these facilities: -
>
> - symbolic algebra manipulation;
> - symbolic differentiation or integration;

As a consequence of this there have previously not been any mathematics examinations in England that have the allowed the use of a computer algebra system (CAS). As a consequence of this CAS are rarely used in the teaching and learning of mathematics in English schools. This is missing an opportunity to take advantage of the benefits of using CAS. Böhm et al (2004, p127) suggest these include making concepts easier to teach, supporting visualisations, saving time on routine calculations and improving students' perception of mathematics.

In 2008 MEI, in partnership with Texas Instruments, convened a seminar and invited leading experts to discuss 'Computer Algebra Systems in the Mathematics Curriculum'. The main findings of this event were (MEI, 2008, p17):

> An ICT-based qualification where students have access to appropriate devices in the classroom and examinations would be useful. It could be a much more realistic qualification that allowed them to be better problem-solvers and mathematicians.

In this context MEI wanted to drive the debate forward by exploring the possibility of having part of the A level study involving the use of technology, including CAS, in a way that its use would be expected in the assessment and consequently this would drive its use in the teaching and learning. The aim of this is to aid the evolution of the role of technology in the curriculum from a computational tool to one that allows for observing and conjecturing as part of the mathematical process, as identified by Trouche (2004). MEI approached OCR (the examination board) who gave their full support to the development of a new unit in the A level Further Mathematics options.

Mathematical Content of the unit

It was decided that the mathematical content of the unit would be taken from pure mathematics. This decision was taken because the current qualifications framework in England favours assessment by examination and it was believed that the study of pure mathematics offered more potential for examinations with technology than applied mathematics, where technology is more useful for project-based continuous assessment. This potential for using technology, and in particular CAS, for pure mathematics at A level was identified by Monaghan (2000) in highlighting its potential to increase the use of parameters and focus more on conceptual ideas and interpretation.

The requirements of regulation also impacted on the choice of topics: it was a requirement that the topics contained mathematics that wasn't assessed elsewhere in A level Mathematics or Further Mathematics. As there are a lot of optional units in A level Mathematics and Further Mathematics this limited the scope of topics for inclusion. The main criterion for inclusion was that they had to be areas of pure mathematics that benefitted from students being able to use tools to access a large number of results quickly and then to be able to analyse these results using their mathematical skills.

The topics chosen were:

- Investigation of curves;
- Functions of complex variables;
- Number Theory.

The investigation of curves can be greatly aided by access to a graph plotter which allows students to plot a family of related curves using parameters which can be changed dynamically. The use of CAS is also helpful in analysing the properties of the curves. Although students meet complex numbers in A level Further Mathematics they do not consider functions of complex variables, such as $\sin(z)$, or polynomials with complex coefficients. Using CAS allows them to investigate these and this is complemented by the use of a spreadsheet for iterations using complex numbers. Number theory is not studied at all in A level Mathematics or Further Mathematics; the use of a programming language, coupled with the access to CAS functions, makes this topic accessible to students at this level.

The full specification of the syllabus detailing the mathematical content can be found via the MEI website (http://www.mei.org.uk/fpt).

Technology

The choice of the topics above suggested that it was necessary for the students to have access to:

- Graph-plotter
- Spreadsheet
- Computer Algebra System
- Programming language

There are different pieces of software that these applications; however, as they will be used together, especially with CAS being used in all the topics, it is advantageous to use a piece of software that features all of these in a way that objects, such as functions, can be used across multiple applications.

There was a conflict in considering which software to use. The examination board were reluctant to require a specific piece of technology as they perceived this ran counter to their aims. Similarly an overt focus on one specific piece of software could result in the unit being perceived as training in using the software as opposed to learning the mathematics. In contrast to this is the concern that many teachers will be unfamiliar with some of the applications used and will desire teaching and learning materials that inform them how to use the software as well as covering the mathematics. It is also essential when designing both the syllabus and the assessment to know the capabilities of the software. Artigue (2002) identifies the complexity of the instrumentation process in which the technology becomes a tool that students can use effectively in mathematical tasks. This process is necessary; however, the goal is for students to learn the mathematics so having a single technology to use reduces the time required for students to go through the instrumentation process.

The compromise reached was that the specifications and features of the software allowed in the examination would be stated by the examination board and schools would be encouraged to seek advice from MEI about which software to use. TI-Nspire was chosen as the piece of software in which to write the teaching and learning resources as it features all the applications in a linked way, it is affordable for schools, and Texas Instruments have a strong record in supporting the use of technology in school mathematics. Schools are likely to be using other software in the teaching and learning and they may choose to prepare students for the examination using other software in future years.

Assessment

To fit in with the current qualifications framework it was felt that a unit assessed by examination was more likely to be approved by the regulators. Consequently the assessment will be an unseen, timed, written examination in line with the assessment of other units in A level Mathematics and Further Mathematics. The students will have access to software in the examination and the questions will typically start with an investigation into a problem using the technology with some follow-up questions in which they analyse their results mathematically. The unit will carry the same weight as other units; however, the examination will be slightly longer, two hours as opposed to one and half hours, to take into account that in the programming question students may need to trial and amend the program to ensure that it works. The students will submit a written response to the questions but there is the possibility that, in future years, electronic submission of some or all of their work will be allowed.

As the topics are those in which students will generate results using the technology and then make inferences or deductions based on these this has resulted in questions that are longer than in other units. Monaghan (2000) suggests that problems with the use of CAS are less acute when longer questions are used but raises the concern that, by moving from questions which examine skills to those of a more conceptual nature, the questions may become more difficult. This is countered by questioning the assumption that standard routine questions are a lifeline to students who obtain low pass grades and offers alternative view that they may be 'good thinkers' who perform less well at reproducing algorithms. It will be interesting to observe how students perform on this style of questions and whether the longer questions impact disproportionately on students working at different grades.

The specification was accepted by the regulators and the first examination is scheduled for June 2013. The expectation is that around fifty students from approximately ten schools will take the unit in the first year.

An example question from the specimen paper is shown in figure 1. The full specimen paper can be found via the MEI website (http://www.mei.org.uk/fpt).

3 (i) Create a program to find all the positive integer solutions to $x^2 - y^2 = 1$, with $x \leq 100$, $y \leq 100$. Write out your program in full and list the solutions it gives. **[10marks]**

(ii) Show how the other solutions can be derived from the solution with the smallest x-value. Use each solution to give a rational approximation to $\sqrt{3}$. **[5 marks]**

(iii) Edit your program so that it will find solutions to $x^2 - ny^2 = 1$, where n is a positive integer. Write out the lines of your program that you have changed. Use the edited program to find a rational approximation to $\sqrt{5}$ that is accurate to within 0.1%. **[6marks]**

(iv) Explain why the edited program will not give any results if n is a square number. **[2marks]**

Figure 1: Example examination question from the specimen paper

Supporting the teachers

A key factor in developing the unit was providing sufficient support for the teachers involved. Goulding and Kyriacou (2007) highlight the importance of the teachers being confident users of the technology and the need to make the links between the outputs explicit to the students. As this is an optional unit teachers could choose whether to offer it to their students and it has been evident in working with them that they are teachers who feel positive about the use of technology in mathematics. However, they may not have used the software before or be familiar with the mathematics in the unit and consequently a support package has been designed and implemented with them.

This support has included teacher meetings and training days plus additional online sessions using a Blackboard Collaborate online classroom. These sessions and meetings have covered both the use of the software and the mathematical content. In addition teaching and learning materials are being provided free, to teachers and students, through MEI's online learning environment: Integral Online Resources for Mathematics (http://integralmaths.org/).

IMPLICATIONS FOR FUTURE DEVELOPMENTS

This unit has been developed to promote the effective use of technology in the teaching and learning of mathematics and provide concrete examples of the challenges and choices needed to be made when designing technology-based mathematics curricula and assessment.

Through developing the unit the following have been observed:

- It is possible to develop technology-based mathematics courses within existing qualifications frameworks.
- The decisions made about the choice of topics demonstrate that technology is effective when it allows learners to access mathematical results quickly and efficiently so they can make inferences and deductions based on these.

- When designing curricula and assessments that allow the use of CAS it is important that the emphasis is on learners selecting and using techniques and not the mechanics of implementing them.
- Although different pieces of software have different qualities it is possible to identify the features that the software should have instead of requiring a specific piece of software: e.g. the graph-plotter should have the ability to control parameters with a slider or scroll-bar.

All of these demonstrate that there are requirements/constraints when developing curricula, but that suitable innovation can produce interesting, exciting and useful developments.

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