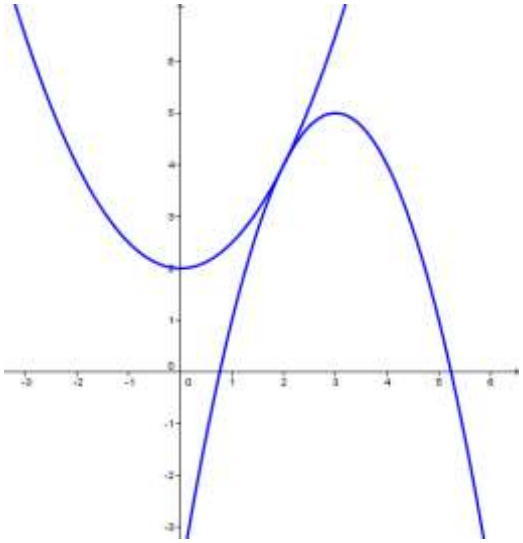
Mathematical Problem Solving
AS/A Level example

Example 9

A class has been studying differentiation and coordinate geometry and the teacher sets the following problem to test both their problem solving skills and their ability to apply the skills they have been taught.

> **Work with quadratic functions and their graphs; the discriminant of a quadratic function, including the conditions for real and repeated roots; completing the square; solution of quadratic equations including solving quadratic equations in a function of the unknown**
>
> **Differentiate x^n, for rational values of n, and related constant multiples, sums and differences**
>
> **Apply differentiation to find gradients, tangents and normals, maxima and minima and stationary points**

DfE: Mathematics GCE AS and A level subject content July 2014



These two quadratic curves have a common tangent at $x = 2$.

Given that one curve has a vertex at $(0,2)$ and the other has a vertex at $(3,5)$, find the equation of each curve and the equation of the common tangent.

Possible questions and prompts

Problem	Stimulus questions	Additional questions	Direct prompt
The student does not know where to start.	What information are you given about each curve in the question? What do you notice about the curve on the left?	What ways are there to write the equation of a quadratic curve? Which ones are the most appropriate for each curve? If it has a vertex at $(0,2)$, what should the equation of that curve look like?	Write down the main features of each of the quadratic curves. Write down an equation for each curve that gives them those features. Use parameters where you do not know the actual values.
The student has introduced too many parameters and is getting bogged down with the algebra.	Have you used the best form for each of the equations of the curve? Could you write the equation of this curve using only one parameter? (this could be applied to either of the curves)	How could you write the equation of this curve using what you know about its vertex? How can you make sure it is the correct way up?	You are going to form some equations using what is the same about both curves.
The student has written an equation for each curve but does not know how to find the parameters.	What information given in the question have you not used yet? What can you say about the y coordinate of each graph when $x = 2$?	Can you write an equation connecting your two parameters using this? You now have an equation with two unknowns, if you want to work out what they are, what will you need?	The two graphs must both pass through the same point when $x = 2$.

Problem	Stimulus questions	Additional questions	Direct prompt
At this stage there are (at least) two ways to continue, one that uses differentiation (which is probably quicker) and one that uses a property of the discriminant of a quadratic equation.			
1. Using calculus			
The student has identified that the curves have the same gradient but does not know how to continue.	How do you find the gradient of a curve? Can you differentiate this quadratic expression with respect to x? (for either expression) Can you write an expression for the gradient at $x = 2$? You have an expression for the gradient of each curve at $x = 2$. What do you know about these two expressions from the information given in the problem?	Can you multiply this out to find an expression that you can differentiate? (If the chain rule hasn't been covered at this stage) Now you have two equations in two unknowns, what will you do?	Differentiation will come in handy here.
2. Forming a quadratic equation and using the discriminant.			
The student has identified that the curves have the same gradient but does not know how to continue.	If the curves have a common tangent at the point where $x = 2$, what does that tell you about how they meet?	Can you find an equation to calculate the x coordinate of where the curves meet (even though you know it is at $x = 2$)? If you solve this equation, how many solutions should it have?	

Problem	Stimulus questions	Additional questions	Direct prompt
		What condition has to be met for it to have only one solution? Can you create another solution? NOTE: There is another highly effective approach here. The student may write out their quadratic expression and use the fact that it is equivalent to $(x-2)^2$.	
Finishing the problem			
The student has both parameters and therefore both of the curve equations but does not know how to find the equation of the tangent.	How can you calculate the gradient of this curve at $x = 2$? What is the y value of this curve at $x = 2$? How do you use the gradient of a straight line and a point it passes through to find the equation of that line?		