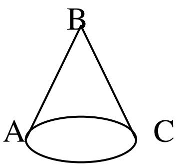
Introduction to matrices

Notes and Examples

These notes contain subsections on
- Matrices
- Multiplying matrices
- Properties of matrix multiplication

Matrices

A matrix is simply a way of storing information. For example, the diagram below shows a map of the roads linking three towns A, B and C. The corresponding '*direct route*' matrix is shown beside it.



$$\begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix} \end{array}$$

In this section you learn to add and subtract matrices, to multiply a matrix by a number and to multiply two matrices.

Matrices are classified by number of rows and the number of columns they have. The matrix above has 3 rows and 3 columns, it is a 3×3 matrix (read as '3 by 3').

A matrix with m rows and n columns is an $m\times n$ matrix. This is called the order of the matrix.

A **square matrix** is a matrix with the same number of rows as columns.

You can add or subtract matrices provided they have the **same order**.

Example 1

$\mathbf{A}$ is the matrix $\begin{pmatrix} 2 & 3 \\ -1 & 0 \end{pmatrix}$, $\mathbf{B}$ is the matrix $\begin{pmatrix} -3 & 4 \\ 1 & -2 \end{pmatrix}$.

Find
(i) $\mathbf{A}+\mathbf{B}$
(ii) $\mathbf{A}-\mathbf{B}$
(iii) $2\mathbf{A}$
(iv) $3\mathbf{B}-\mathbf{A}$

Solution

(i) $\mathbf{A}+\mathbf{B}=\begin{pmatrix}2 & 3\\ -1 & 0\end{pmatrix}+\begin{pmatrix}-3 & 4\\ 1 & -2\end{pmatrix}=\begin{pmatrix}-1 & 7\\ 0 & -2\end{pmatrix}$

(ii) $\mathbf{A}-\mathbf{B}=\begin{pmatrix}2 & 3\\ -1 & 0\end{pmatrix}-\begin{pmatrix}-3 & 4\\ 1 & -2\end{pmatrix}=\begin{pmatrix}5 & -1\\ -2 & 2\end{pmatrix}$

(iii) $2\mathbf{A}=2\begin{pmatrix}2 & 3\\ -1 & 0\end{pmatrix}=\begin{pmatrix}4 & 6\\ -2 & 0\end{pmatrix}$

(iv) $3\mathbf{B}-\mathbf{A}=3\begin{pmatrix}-3 & 4\\ 1 & -2\end{pmatrix}-\begin{pmatrix}2 & 3\\ -1 & 0\end{pmatrix}=\begin{pmatrix}-9 & 12\\ 3 & -6\end{pmatrix}-\begin{pmatrix}2 & 3\\ -1 & 0\end{pmatrix}=\begin{pmatrix}-11 & 9\\ 4 & -6\end{pmatrix}$

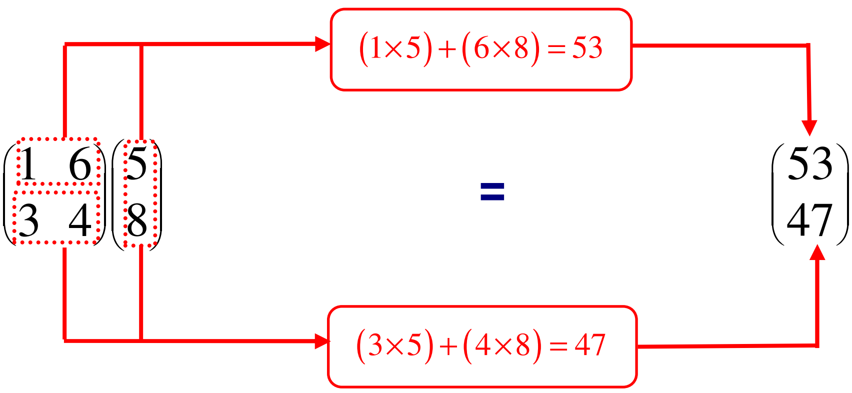
Multiplying matrices

Multiplying matrices is an important skill which you must master. It takes a bit of getting used to, but after plenty of practice you will find it quite straightforward.

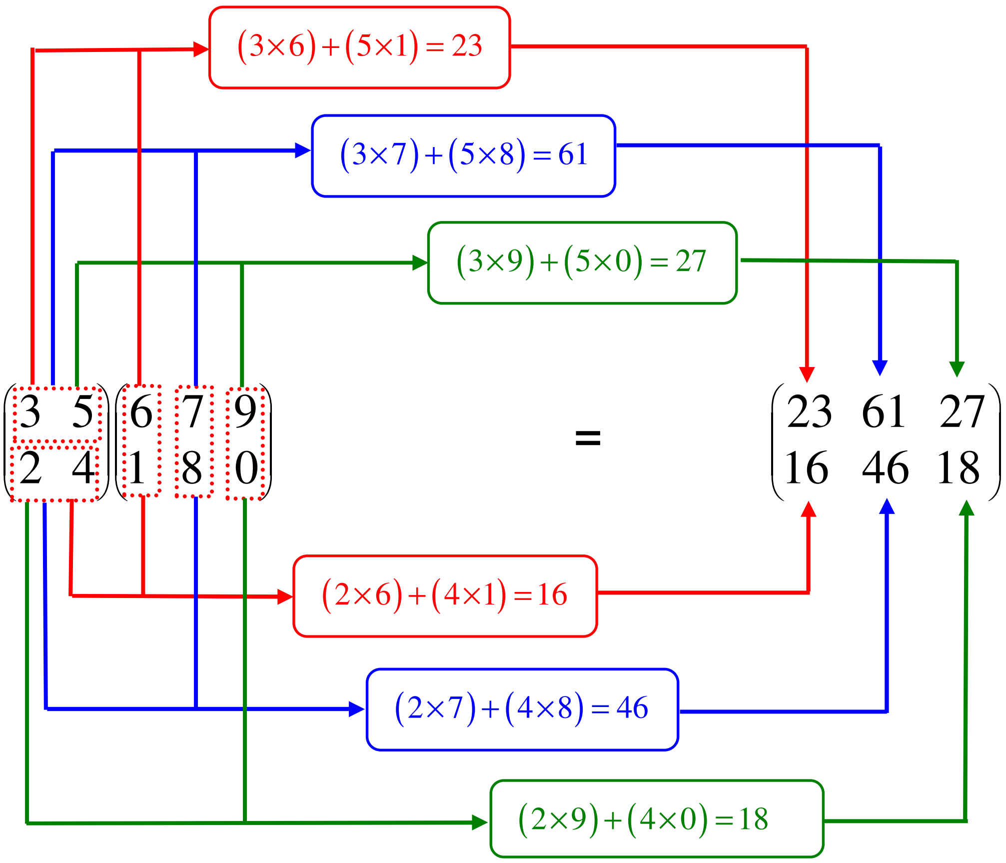
The important points to remember are:

- Use each row of the first matrix with each column of the second.
- When you are using row a of the first matrix with column b of the second matrix, the result gives you the element in row a, column b of the product matrix.
- To multiply matrices, the number of columns in the first matrix must be the same as the number of rows in the second matrix. If this is not the case, the matrices do not conform and cannot be multiplied.

The diagram below shows the steps used when carrying out the multiplication $\begin{pmatrix}1 & 6\\ 3 & 4\end{pmatrix}\begin{pmatrix}5\\ 8\end{pmatrix}$.



A similar technique applies to all matrix multiplications. You use each row of the first (i.e. left) matrix with each column, in turn, of the second matrix. The diagram below shows the steps used when multiplying a 2×2 matrix by a 2×3 matrix. The product is another 2×3 matrix.



If you multiply a 3×4 matrix (on the left) by a 4×2 matrix (on the right) similar rules apply: the product is a 3×2 matrix. For example:

$$\begin{pmatrix} 1 & 2 & 4 & 7 \\ -3 & 5 & 0 & 1 \\ 4 & 2 & 3 & 5 \end{pmatrix}\begin{pmatrix} 5 & 1 \\ -6 & 4 \\ 8 & 9 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 39 & 59 \\ -43 & 19 \\ 42 & 49 \end{pmatrix}$$

Example 2

A is the matrix $\begin{pmatrix} 2 & 3 \\ -1 & 5 \end{pmatrix}$.

B is the matrix $\begin{pmatrix} -3 & 2 & 0 \\ 1 & 4 & -2 \end{pmatrix}$.

C is the matrix $\begin{pmatrix} 1 & -1 \\ 0 & 3 \\ 2 & 4 \end{pmatrix}$.

Find where possible

(i)	**AB**	(ii)	**BA**	(iii)	**BC**
(iv)	**CB**	(v)	**AC**	(vi)	**CA**

Solution

(i) **A** is a 2×2 matrix and **B** is a 2×3 matrix, so these matrices conform.

$$\mathbf{AB} = \begin{pmatrix} 2 & 3 \\ -1 & 5 \end{pmatrix}\begin{pmatrix} -3 & 2 & 0 \\ 1 & 4 & -2 \end{pmatrix} = \begin{pmatrix} 2\times-3+3\times1 & 2\times2+3\times4 & 2\times0+3\times-2 \\ -1\times-3+5\times1 & -1\times2+5\times4 & -1\times0+5\times-2 \end{pmatrix}$$

$$= \begin{pmatrix} -3 & 16 & -6 \\ 8 & 18 & -10 \end{pmatrix}$$

(ii) **B** is a 2 × 3 matrix and **A** is a 2 × 2 matrix, so these matrices do not conform. It is not possible to find the product **BA**.

(iii) **B** is a 2 × 3 matrix and **C** is a 3 × 2 matrix, so these matrices conform.

$$\mathbf{BC} = \begin{pmatrix} -3 & 2 & 0 \\ 1 & 4 & -2 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 3 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} -3\times1+2\times0+0\times2 & -3\times-1+2\times3+0\times4 \\ 1\times1+4\times0+-2\times2 & 1\times-1+4\times3+-2\times4 \end{pmatrix}$$

$$= \begin{pmatrix} -3 & 9 \\ -3 & 3 \end{pmatrix}$$

(iv) **C** is a 3 × 2 matrix and **B** is a 2 × 3 matrix, so these matrices conform.

$$\mathbf{CB} = \begin{pmatrix} 1 & -1 \\ 0 & 3 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} -3 & 2 & 0 \\ 1 & 4 & -2 \end{pmatrix} = \begin{pmatrix} 1\times-3+-1\times1 & 1\times2+-1\times4 & 1\times0+-1\times-2 \\ 0\times-3+3\times1 & 0\times2+3\times4 & 0\times0+3\times-2 \\ 2\times-3+4\times1 & 2\times2+4\times4 & 2\times0+4\times-2 \end{pmatrix}$$

$$= \begin{pmatrix} -4 & -2 & 2 \\ 3 & 12 & -6 \\ -2 & 20 & -8 \end{pmatrix}$$

(v) **A** is a 2 × 2 matrix and **C** is a 3 × 2 matrix, so these matrices do not conform. It is not possible to find the product **AC**.

(vi) **C** is a 3 × 2 matrix and **A** is a 2 × 2 matrix, so these matrices conform.

$$\mathbf{CA} = \begin{pmatrix} 1 & -1 \\ 0 & 3 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ -1 & 5 \end{pmatrix} = \begin{pmatrix} 1\times2+-1\times-1 & 1\times3+-1\times5 \\ 0\times2+3\times-1 & 0\times3+3\times5 \\ 2\times2+4\times-1 & 2\times3+4\times5 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & -2 \\ -3 & 15 \\ 0 & 26 \end{pmatrix}$$

Properties of matrix multiplication

Make sure that you know the important properties of matrix multiplication:

- Matrices must be conformable for multiplication
- Matrix multiplication is not commutative
- Matrix multiplication is associative
- Matrix multiplication is distributive

You have already seen in Example 2 above that matrix multiplication is not commutative. In that case, **AB** exists but **BA** does not, **BC** and **CB** both exist but are different (in fact they have different orders) and **AC** does not exist but **CA** does.

Example 3 proves that matrix multiplication is associative for any 2 × 2 matrices.

Example 3

Using $\mathbf{P}=\begin{pmatrix} a & c \\ b & d \end{pmatrix}$, $\mathbf{Q}=\begin{pmatrix} e & g \\ f & h \end{pmatrix}$ and $\mathbf{R}=\begin{pmatrix} i & k \\ j & l \end{pmatrix}$, find

(i) **PQ** (ii) **(PQ)R** (iii) **QR** (iv) **P(QR)**

and so demonstrate that matrix multiplication is associative.

Solution

(i) $\mathbf{PQ}=\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} e & g \\ f & h \end{pmatrix}=\begin{pmatrix} ae+cf & ag+ch \\ be+df & bg+dh \end{pmatrix}$

(ii) $(\mathbf{PQ})\mathbf{R}=\begin{pmatrix} ae+cf & ag+ch \\ be+df & bg+dh \end{pmatrix}\begin{pmatrix} i & k \\ j & l \end{pmatrix}$

$=\begin{pmatrix} aei+cfi+agj+chj & aek+cfk+agl+chl \\ bei+dfi+bgj+dhj & bek+dfk+bgl+dhl \end{pmatrix}$

(iii) $\mathbf{QR}=\begin{pmatrix} e & g \\ f & h \end{pmatrix}\begin{pmatrix} i & k \\ j & l \end{pmatrix}=\begin{pmatrix} ei+gj & ek+gl \\ fi+hj & fk+hl \end{pmatrix}$

(iv) $\mathbf{P}(\mathbf{QR})=\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} ei+gj & ek+gl \\ fi+hj & fk+hl \end{pmatrix}$

$=\begin{pmatrix} aei+agj+cfi+chj & aek+agl+cfk+chl \\ bei+bgj+dfi+dhj & bek+bgl+dfk+dhl \end{pmatrix}$

(PQ)R = P(QR) so matrix multiplication is associative for all 2 × 2 matrices.

You could carry out a similar proof for matrices of any order, provided they were conformable, i.e. their orders were $p \times q$, $q \times r$ and $r \times s$ respectively.

Example 4 proves that matrix multiplication is distributive for any 2 × 2 matrices.

Example 4

Using $\mathbf{P}=\begin{pmatrix} a & c \\ b & d \end{pmatrix}$, $\mathbf{Q}=\begin{pmatrix} e & g \\ f & h \end{pmatrix}$ and $\mathbf{R}=\begin{pmatrix} i & k \\ j & l \end{pmatrix}$, find

(i) **P(Q + R)** (ii) **PQ + PR** (iii) **(P + Q)R** (iv) **PR + QR**

and so demonstrate the distributive property of matrix multiplication over matrix addition.

Solution

(i) $\mathbf{P}(\mathbf{Q}+\mathbf{R}) = \begin{pmatrix} a & c \\ b & d \end{pmatrix}\left(\begin{pmatrix} e & g \\ f & h \end{pmatrix} + \begin{pmatrix} i & k \\ j & l \end{pmatrix}\right)$

$$= \begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} e+i & g+k \\ f+j & h+l \end{pmatrix}$$

$$= \begin{pmatrix} a(e+i)+c(f+j) & a(g+k)+c(h+l) \\ b(e+i)+d(f+j) & b(g+k)+d(h+l) \end{pmatrix}$$

(ii) $\mathbf{PQ}+\mathbf{PR} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} e & g \\ f & h \end{pmatrix} + \begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} i & k \\ j & l \end{pmatrix}$

$$= \begin{pmatrix} ae+cf & ag+ch \\ be+df & bg+dh \end{pmatrix} + \begin{pmatrix} ai+cj & ak+cl \\ bi+dj & bk+dl \end{pmatrix}$$

$$= \begin{pmatrix} ae+cf+ai+cj & ag+ch+ak+cl \\ be+df+bi+dj & bg+dh+bk+dl \end{pmatrix}$$

(iii) $(\mathbf{P}+\mathbf{Q})\mathbf{R} = \left(\begin{pmatrix} a & c \\ b & d \end{pmatrix} + \begin{pmatrix} e & g \\ f & h \end{pmatrix}\right)\begin{pmatrix} i & k \\ j & l \end{pmatrix}$

$$= \begin{pmatrix} a+e & c+g \\ b+f & d+h \end{pmatrix}\begin{pmatrix} i & k \\ j & l \end{pmatrix}$$

$$= \begin{pmatrix} (a+e)i+(c+g)j & (a+e)k+(c+g)l \\ (b+f)i+(d+h)j & (b+f)k+(d+h)l \end{pmatrix}$$

(iv) $\mathbf{PR}+\mathbf{QR} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} i & k \\ j & l \end{pmatrix} + \begin{pmatrix} e & g \\ f & h \end{pmatrix}\begin{pmatrix} i & k \\ j & l \end{pmatrix}$

$$= \begin{pmatrix} ai+cj & ak+cl \\ bi+dj & bk+dl \end{pmatrix} + \begin{pmatrix} ei+gj & ek+gl \\ fi+hj & fk+hl \end{pmatrix}$$

$$= \begin{pmatrix} ai+cj+ei+gj & ak+cl+ek+gl \\ bi+dj+fi+hj & bk+dl+fk+hl \end{pmatrix}$$

So $\mathbf{P}(\mathbf{Q}+\mathbf{R}) = \mathbf{PQ}+\mathbf{PR}$ and $(\mathbf{P}+\mathbf{Q})\mathbf{R} = \mathbf{PR}+\mathbf{QR}$

The identity matrix

The matrix **I** = $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is called the 2×2 **identity matrix** because when you multiply any 2×2 matrix **A** by **I** you get **A** as the answer.

I acts like the number 1 in the multiplication of numbers.

This means that for any 2×2 matrix **A**:

IA = **AI** = **A.**

Example 5

A is the matrix $\begin{pmatrix} 2 & 3 \\ -1 & 5 \end{pmatrix}$.

Find the matrix **B** such that **AB** = **I.**

Solution

Firstly for the product matrix to be 2×2, matrix **B** must also be 2×2.

Let **B** = $\begin{pmatrix} \boldsymbol{a} & \boldsymbol{b} \\ \boldsymbol{c} & \boldsymbol{d} \end{pmatrix}$ and you need to find *a, b, c* and *d*.

Now **AB** = **I**

$$\Rightarrow \quad \begin{pmatrix} 2 & 3 \\ -1 & 5 \end{pmatrix}\begin{pmatrix} \boldsymbol{a} & \boldsymbol{b} \\ \boldsymbol{c} & \boldsymbol{d} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Multiplying out the left-hand side gives:

$$\begin{pmatrix} 2\boldsymbol{a}+3\boldsymbol{c} & 2\boldsymbol{b}+3\boldsymbol{d} \\ -\boldsymbol{a}+5\boldsymbol{c} & -\boldsymbol{b}+5\boldsymbol{d} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Equating terms in the first column of each side gives:

$$2a+3c = 1$$
$$-a+5c = 0$$

Solving these equations simultaneously gives $a=\frac{5}{13}$ and $c=\frac{1}{13}$.

Equating terms in the second column of each side gives:

$$2b+3d=0$$
$$-b+5d=1$$

Solving these equations simultaneously gives $b=-\frac{3}{13}$ and $d=\frac{2}{13}$.

So matrix **B** = $\begin{pmatrix} \frac{5}{13} & -\frac{3}{13} \\ \frac{1}{13} & \frac{2}{13} \end{pmatrix}$.