"Big Ideas" **Constrained Optimisation and Computer-based Examining at A level.**
(and don't forget simulation)

A revisiting of:

Computer-based examining in GCE (A level) Decision and Discrete Mathematics
Teaching Mathematics and its Applications
Volume 20, number 2, 2001
Pp. 89-97
See http://teamat.oxfordjournals.org/cgi/content/abstract/20/2/89

Keith Parramore, University of Brighton,
Chief Examiner, OCR

1. Introduction

The article was published in 2001. It detailed the background and development of a module in which students use a computer in the examination – not to facilitate the examination, as happens in some multiple-choice tests, but to do the mathematics.

This is standard practice in HE in many disciplines, notably in statistics, where for more than 20 years students have been analyzing large data sets in examinations. In developing the original experimental trial it was envisaged that it would beat a path for others, particularly statistics, to follow. That has not happened – yet.

The module, originally 2622 and now 4773, has been examined 8 times since 2000, and a wealth of interesting material is now available. The numbers entering for each year are listed below.

Year	2000	2001	2002	2003	2004	2005	2006	2007
Entries	68	53	86	118	170	105	74	64

What follows is a review of the original article, together with a few example questions and a brief examination of where we now are.

2. What is (Decision and) Discrete Mathematics ?

We have moved on in some respects since 2001. Our area is now "Decision Mathematics", which better describes it. There are moves afoot to re-brand it as "Operational Research". There are arguments for and against this – we shall see.

3. Genesis

In this section the background of Decision and Discrete Mathematics was rehearsed. Nothing has changed there!

4. The rationale for Discrete Mathematics at A level

The essence of this section was two points:

- Decision Maths does not require students to have mastery of a large range of algebraic skills. Furthermore the problems seem to have direct relevance to aspects of students' lives – how to organise a drama production; how to plan a delivery route; how to optimise a queuing system.

- Accessibility does not equate with ease. The focus is on modelling, and modelling is hard! Developing modelling skills is a valuable, and valued, thing to do.

Again – no change!

5. Assessment issues

We have moved on since 2001. We have developed examining skills to the extent that we are now confident that we can assess modelling within the confines of a timed, written examination. This is just as well considering the widespread abuse of coursework. Too many "recipes" became available for candidates, and too much internet "help" undermined the process.

6. The MEI Decision and Discrete Computation syllabus

Optimisation was identified as a central theme and unifying concept. It had originally been perceived that computer-based work on optimization would permeate the entire suite of modules. The practicalities of the time forced a reining-back of that.

However, DC is assessed entirely by the computer-based examination. In the examination, candidates submit written answers to questions in the usual way, but are also be required to submit computer print-outs of their results.

Note: The Qualifications and Curriculum Agency made a grant of £5000 available in the year to April 1999 to help with the work of developing these ICT based modules (Decision + Numerical) for trialling purposes. This phase of the project was supported by OCR and the ITAL Unit of UCLES Research and Evaluation Division.

I make no apology for reproducing in full the next four sections of the original paper. Above all else Decision Maths is about developing modelling ability through some nice mathematics, and sections 7, 8 and 10 were there to give a flavour of how this is done in D2C.

Section 7 posed a challenge based on a problem being experienced at the time at the time by the subject officer. It was shown therein that the problem is a constrained optimization problem, and a single example was solved using Lindo. But the subject officer needed to solve it for many candidates. After working for some time to develop my VBA (Visual Basic for Applications) skills I can now do that using Excel. This is interesting since Excel's optimizer "Solver" is nonlinear quasi-Newton/conjugate direction search routine. Such routines find local optima in multivariate constrained optimization problems, whereas, if the problem is linear, then LP guarantees the global optimum. However, Solver can be "tweaked" to behave as a linear optimizer, and can be incorporated (not without difficulty) into a VBA program (see http://www.cmis.brighton.ac.uk/staff/kp4/BigIdeas/BigIdeas.xls).
(If accessing through a server then "save" first – do not open from within the browser.)

Section 8 tried to give some feeling for the wide range of applicability of linear programming.

Section 9 reported on the research being conducted at the time. The report on this can be found at ***.

---

7. A paradigm – Linear Programming

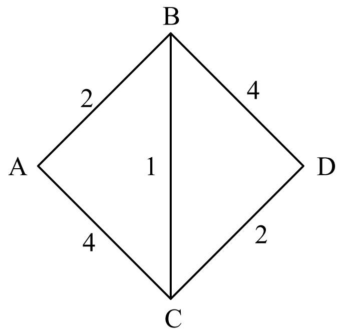
First a problem! Suppose that an A level candidate in a new specification has marks in 15 units and presents them for cashing in for two A levels. Consider the problem of awarding the best possible pair of grades, forgetting, in the best tradition of mathematical modelling, all of the constraints on AS and A2 units, etc. We can easily choose the best 6 units and thus decide what will be the candidate's best grade. But we must then start again and choose 6 units to give that best grade whilst ensuring that the remaining units give the best possible second grade. The complete enumeration involves an impractical 420420 possibilities – ${}^{15}C_6 \times {}^{9}C_6$. But the problem can be specified as an integer linear programming problem, and we show a formulation in section 10. In first trying the problem for him or herself the reader will be able to judge the potential for avoiding the stereotypical in Discrete Maths, and develop some appreciation of the level of difficulty at which questions can be pitched.

First we take a brief tour of some of the modelling possibilities that can be explored, given an LP package. The word "given" is appropriate, for decently powerful LP packages **are** given away (see www.lindo.com).

8. Some LP models

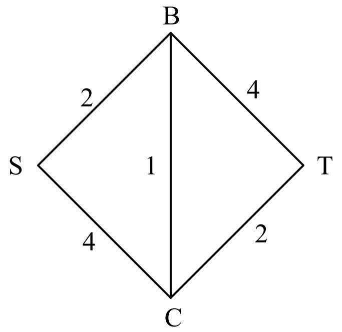
The following LPs are intended to give some idea of the ubiquity of LP. The problems which they tackle are tiny, and are used only to illustrate the approach. Explanations are not furnished, the reader being invited to interpret the objective and the constraints.

A shortest path problem:




```
MIN 2AB+4BD+4AC+2CD+BC+CB
ST  AB+AC=1
    AB+CB-BC-BD=0
    AC+BC-CB-CD=0
    BD+CD=1
END
```

A network flow problem:




```
MAX  SB+SC
ST   SB+CB-BC-BT=0
     SC+BC-CB-CT=0
     SB<=2
     BT<=4
     SC<=4
     CT<=2
     BC<=1
     CB<=1
END
```

A matching problem:

	1	2	3	4
A	x			x
B	x		x	
C		x	x	
D			x	

"x"s indicate possibilities

```
MAX  A1+A4+B1+B3+C2+C3+D3
ST   A1+A4<=1
     B1+B3<=1
     C2+C3<=1
     D3<=1
     A1+B1<=1
     C2<=1
     B3+C3+D3<=1
     A4<=1
END
```

An allocation problem:

	1	2	3	4
A	5	2	3	6
B	1	7	2	4
C	5	8	3	1
D	4	4	2	6

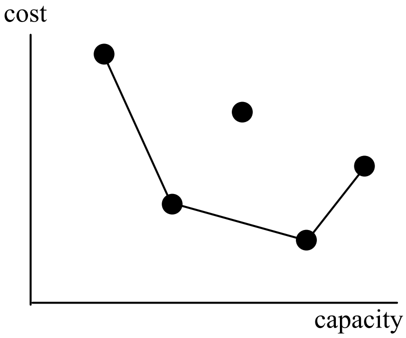
```
MIN 5A1+2A2+3A3+6A4+B1+7B2
    +2B3+4B4+5C1+8C2+3C3+C4
    +4D1+4D2+2D3+6D4
ST  A1+A2+A3+A4=1
    B1+B2+B3+B4=1
    C1+C2+C3+C4=1
    D1+D2+D3+D4=1
    A1+B1+C1+D1=1
    A2+B2+C2+D2=1
    A3+B3+C3+D3=1
    A4+B4+C4+D4=1
END
```

A transportation problem:

		5	5	5	5
		1	2	3	4
3	A	5	2	3	6
6	B	1	7	2	4
9	C	5	8	3	1
2	D	4	4	2	6

```
MIN 5A1+2A2+3A3+6A4+B1+7B2
    +2B3+4B4+5C1+8C2+3C3+C4
    +4D1+4D2+2D3+6D4
ST  A1+A2+A3+A4=3
    B1+B2+B3+B4=6
    C1+C2+C3+C4=9
    D1+D2+D3+D4=2
    A1+B1+C1+D1=5
    A2+B2+C2+D2=5
    A3+B3+C3+D3=5
    A4+B4+C4+D4=5
END
```

An envelope problem:




```
MIN 8λ1+3λ2+6λ3+2λ4+4λ5
ST  λ1+2λ2+3λ3+4λ4+5λ5–X = 0
    λ1<=1
    λ2<=1
    λ3<=1
    λ4<=1
    λ5<=1
    λ1+λ2+λ3+λ4+λ5>=1
    X = 3.8
END
```

The final problem is too opaque to leave without **any** explanation. It occurred as the solution to a practical problem involving the minimisation of pumping costs. The pumps have capacities of 1, 2, 3, 4 and 5 units per day. Their costs are 8, 3, 6, 2 and 4 per day respectively. Pumps can be employed for part of a day (the λs) and at least one pump must be in operation at all times. The amount that is required is set to 3.8 in the example. The solution to the LP gives the cheapest combination of pumps to deliver the required amount.

9. The Field Trials

Field trials of the Decision and Discrete Computation and Numerical Computation units were carried out in 2000 and 2001 at self-selected schools and colleges. It was supported by OCR and their ITAL Unit, UCLES Research and Evaluation Division. The research was aimed both at assessment and educational issues in the following areas.

<u>Test Administration Issues</u>

- Practical issues concerning machine availability, confidentiality, et al.
- Software – Excel and LINDO
- Practical issues concerned with the running of the examination

<u>Assessment and Educational Issues</u>

- The efficacy of the examination
- Questionnaire-based research on teachers' and students' perceptions

10. Answer (to the grading problem)

The grading problem (from section 7) first:

Let $m_1, m_2, m_3, \ldots, m_{15}$ be the marks on the 15 units.

Let M be the minimum mark which achieves the grade determined by the best 6 of the marks.

Let $x_1, x_2, x_3, \ldots, x_{15}$ be indicator variables, with $x_i = 1$ if unit number i is chosen for the first A level, and 0 otherwise.

Let $y_1, y_2, y_3, \ldots, y_{15}$ be indicator variables, with $y_i = 1$ if unit number i is chosen for the second A level, and 0 otherwise.

Then the grading problem can be formulated (in LINDO syntax) as:

MAX $\sum_{i=1}^{15} y_i m_i$ — chooses the best for the second A level

ST $\sum_{i=1}^{15} x_i m_i \geq M$ — ensures that the first A level has the required grade

$\sum_{i=1}^{15} x_i = 6$ — ensures that the first A level has 6 units

$\sum_{i=1}^{15} y_i = 6$ — ensures that the second A level has 6 units

$x_i + y_i \leq 1$ for each of i=1, i=2, …, i=15 — fifteen constraints, ensuring that a unit can be used at most once

END

INT 30 — defines the variables to be 0/1 indicator variables

LINDO will solve this problem efficiently using branch and bound. A trial application solved the problem almost instantly in 40 steps.

This problem is illustrative of what can be set, with appropriate guidance, as an examination question.

11. Conclusion

The original conclusion included:

> *Whilst computer-based testing has been around for some time, the use of computers as a tool within the subject area has not yet been seen in public examinations. It may be argued that such use is inevitable. This scheme, independent as it is of such wider issues, may help to focus the arguments.*

Over the following years the material has proved to be accessible to A level students – its educational value was never in doubt. See http://www.mei.org.uk/meiresources/apapers.shtml for past papers. (The password is *Elephant*.)

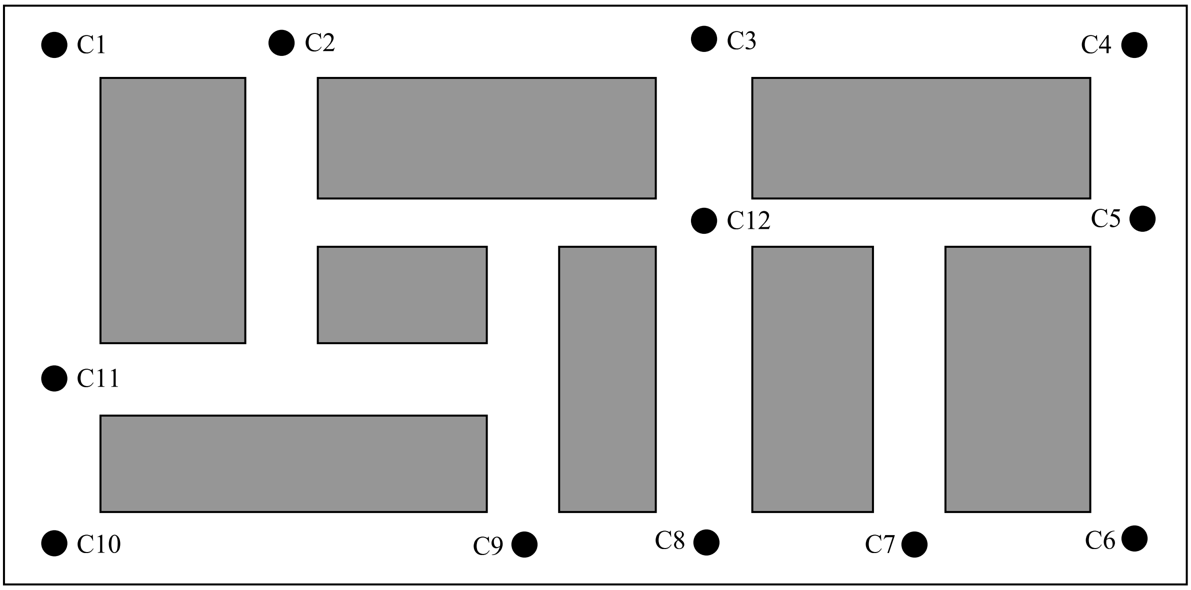
Furthermore, the methodology of using a computer-based examination has been implemented with no significant problems. The worst that can be said is that the papers are more difficult to mark than are conventional mathematics papers, there being more scope for candidate output to vary, and there being more of it!

The disappointment is that there has been little take-up of this specific option, and no take-up of the use of computers in examinations in other areas. Are we just still ahead of the game?

… and if you can't solve it any other way, then there's always simulation!

Two questions and solutions from June 2007.

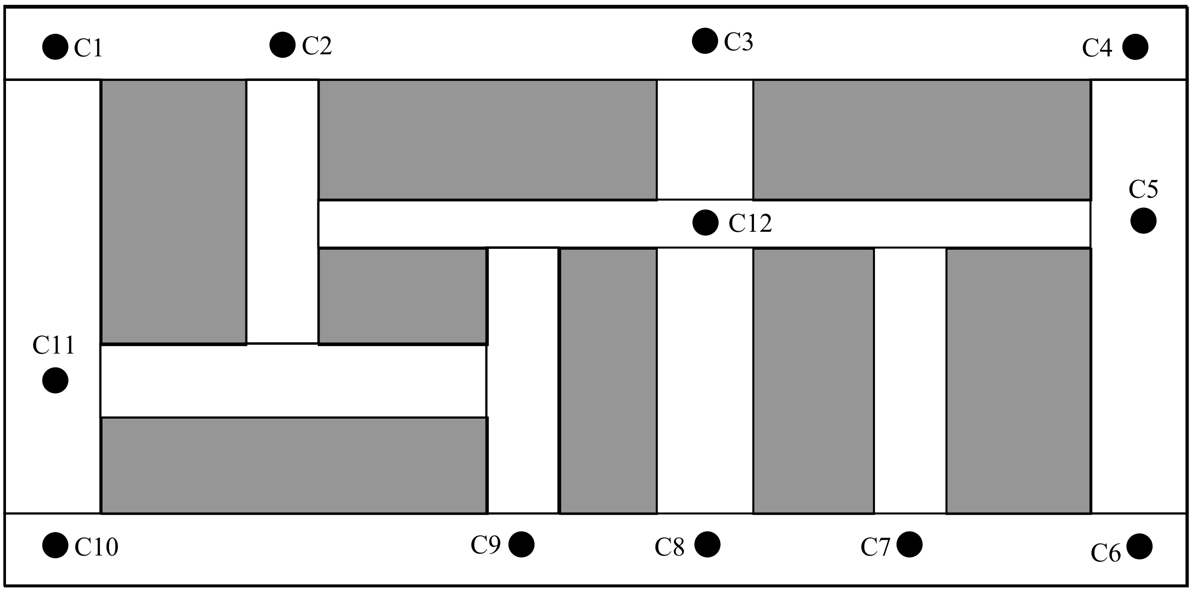
3. The builders of a shopping precinct have to decide where to place CCTV cameras. The diagram shows buildings (which are shaded), pavements, and 12 possible locations for cameras.



Cameras can be rotated to view along different directions, and all pavements must be in sight of at least one camera.

(i) By inspection select a set of 6 locations from which cameras can scan all pavements **2**

The diagram below shows one way of splitting the pavements into rectangles.



(ii) Formulate an LP to select a minimum set of locations from which cameras can scan all of the rectangles. Produce a print-out of your formulation. **8**

(iii) Run your LP, produce a print-out of your output, and interpret the results. **3**

The costs of installing a camera depends on the location. They are listed below.

Location	C1	C2	C3	C4	C5	C6	C7	C8	C9	C10	C11	C12
Cost (£000)	5	2	3	5	4	1.5	2	2	5	3	4	7

(iv) Modify your LP to find the cheapest way of achieving full coverage of all pavements. **2**

(v) Run your modified LP, produce a print-out of your output, and interpret the results. **3**

(i) e.g. C2 C3 C5 C7 C9 C11	M1 A1
(ii)	
Min C1+C2+C3+C4+C5+C6+C7+C8+C9+C10+C11+C12	M1 objective
st C1+C2+C3+C4>=1	A1
C4+C5+C6>=1	
C6+C7+C8+C9+C10>=1	M1
C1+C10+C11>=1	A5 constraints
C2>=1	(−1 each error)
C3+C8+C12>=1	
C5+C12>=1	
C11>=1	
C9>=1	
C7>=1	
end	
(iii)	
LP OPTIMUM FOUND AT STEP 7	B1 running
OBJECTIVE FUNCTION VALUE	
1) 6.000000	
VARIABLE VALUE REDUCED COST	
C1 0.000000 0.000000	
C2 1.000000 0.000000	
C3 0.000000 1.000000	
C4 1.000000 0.000000	
C5 0.000000 0.000000	
C6 0.000000 0.000000	
C7 1.000000 0.000000	
C8 0.000000 1.000000	
C9 1.000000 0.000000	
C10 0.000000 0.000000	
C11 1.000000 0.000000	
C12 1.000000 0.000000	
Use locations 2, 4, 7, 9, 11 and 12.	B1
6 cameras needed	B1
(iv) New objective:	M1
5C1+2C2+3C3+5C4+4C5+1.5C6+2C7+2C8+5C9	A1
+3C10+4C11+7C12	
(v) Running	B1
Use locations 2, 5, 7, 8, 9 and 11.	B1
Cost = £19000	B1

4. A component in a machine has a short lifespan. It fails either after 1, 2 or 3 days, with probabilities given in the table.

Time to failure (days)	1	2	3
Probability	0.1	0.3	0.6

When a component fails it is replaced at the end of the day.

(i) Construct a look-up table to simulate the failure time for a component. Print out the formulae which you use. **3**

(ii) Set up a spreadsheet to simulate failure times for a number of components so that you can accumulate the times to failure. Simulate enough components so that the accumulated failure times exceed 16 days.
Print out your spreadsheet formulae. **3**

(iii) From your simulation in part **(ii)** record

- whether or not there was a failure on day 14,
- whether or not there was a failure on day 15,
- whether or not there was a failure on day 16,
- the total number of failures up to and including day 14.

Repeat your simulation 9 more times (10 times in total), recording information as before. Hence estimate the probability of a failure on day 14, the probability of a failure on day 15, the probability of a failure on day 16, and the expected number of failures up to and including day 14. **5**

Replacing a part when it fails costs £50, plus the cost of the component, which is £25. An alternative policy is to replace a component if it fails on its first day, and otherwise to replace it anyway, failed or not, at the end of its second day. Such a scheduled replacement costs £30 plus the cost of the component.

(iv) Simulate the operation of this scheduled replacement policy over a period of 14 days. Repeat your simulation 10 times and use your results from part (iii) to see whether or not this policy is cost effective. **6**

(v) How could you improve the reliability of your results? **1**

(i) e.g. 1 0 =LOOKUP(RAND(),B1:B3,A1:A3) 2 0.1 3 0.4	B1 rand B1 probs B1 outcomes

(ii) =LOOKUP(RAND(),$B$3:$B$5,$A$3:$A$5)
+ accumulation

e.g.

2	2	3	24
3	5	3	27
2	7	2	29
3	10	2	31
2	12	3	34
3	15	2	36
3	18	2	38
3	21	3	41

M1 formula
A1 repeats
B1 accumulation

(iii) e.g.

day 14	day 15	day16	no. of replacements
0	0	1	5
1	0	0	6
0	0	1	5
0	0	1	5
1	0	0	6
0	1	0	5
0	1	0	6
1	0	1	5
0	1	0	5
1	0	1	6
0.4	0.3	0.5	5.4

M1 first run
A1

B1 repetitions

B1 probabilities
B1 expected no. of replacements

(iv) e.g.

				Replacements day 1	Replacements day 2
1	0	1	1	2	6
2	0.1	1	2	0	7
		2	4	0	7
		2	6	1	6
		2	8	1	6
		2	10	0	7
		2	12	1	6
		2	14	0	7
		1	15	0	7
		2	17	1	6
		2	19		
		2	21	0.6	6.5

B1 changed probabilities
B1 repetitions
B1 results
B1 averages

5.4*(50+25) = 405 versus
0.6*(50+25) + 6.5*(30+25) = 402.5

B1
B1

(v) More repetitions.

B1