Markov Chains: an enrichment topic accessible to all

MEI Conference 2010

MEI

Why does Markov Chains make for great enrichment?

- Straightforward…
- New topics
- Spreadsheets
- Probability in the long term
- Surprising results
- Lots of experimentation possible
- Computer simulation
- Definitions and Proof
- Modern applications that are meaningful to young students

Enrichment Session

- I'll present the material in the session in the same way as if it were for an enrichment session

Founders of Google

Larry Page

Sergey Brin

Students at Stanford University, California whose search engine at the time was nicknamed 'Backrub'.

Google: Number of searches

December 2009:

- **Google:** 88 billion per month
- **Twitter:** 19 billion per month
- **Yahoo:** 9.4 billion per month
- **Bing:** 4.1 billion per month

The Google Empire Today

- Google Docs
- Google Analytics
- Google Earth
- Google Maps
- Google Desktop Search
- Gmail
- Google Image Search

We asked 100 people
- why is Google
the world's most popular
search engine?

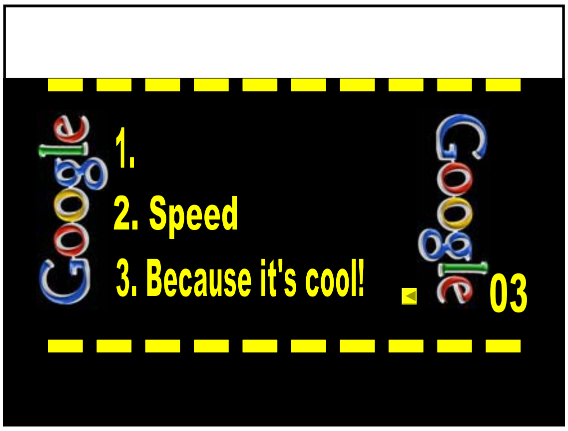
1.
2.
3.
03

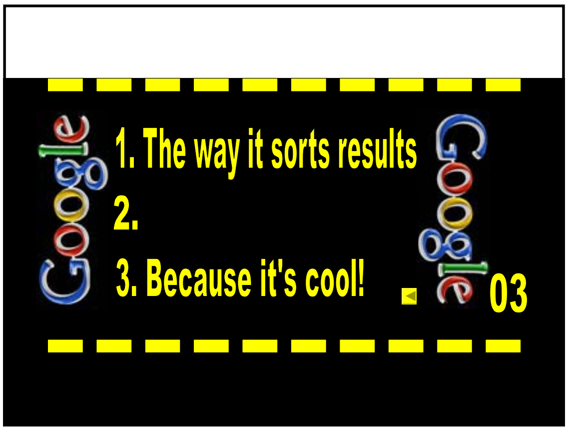
1. The way it sorts results
2.
3.
03

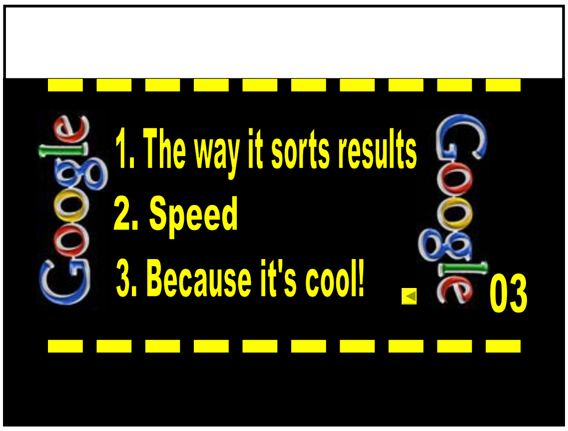
1.
2. Speed
3.
03

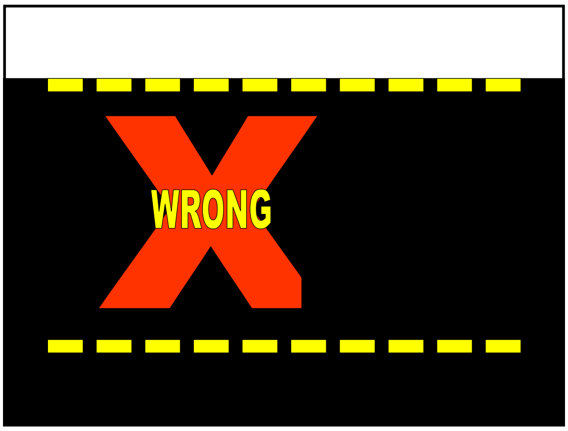
1.
2.
3. Because it's cool!
03

1. The way it sorts results
2. Speed
3.
03









The mathematical secrets of Google

- You're about to learn the mathematical secret that was the basis for Page and Brin's brilliant idea and how this links to Markov Chains.
- This is extremely valuable information..

Matrices and Matrix Multiplication

- Introduction
- Use excel to get the product of two 2x2 matrices and then two 3x3 matrices.

What is a Markov Chain

- It's a system that has a number of states and it moves from state to state with certain probabilities.

Typical First example

- **An island**

If it's sunny today the probability that it will be sunny tomorrow is 0.7 and the probability that it rains is 0.3.

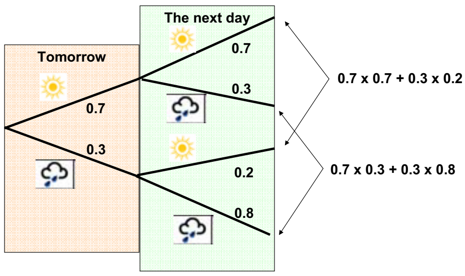
If it rains today the probability that it will rain tomorrow is 0.8 and the probability that it will be sunny is 0.2

Transition Matrix

$$
\textbf{Tomorrow}\quad
\begin{array}{c}
 \\ \text{Sun} \\ \text{Rain}
\end{array}
\overset{\textbf{Today}}{\begin{array}{cc}
\text{Sun} & \text{Rain} \\
\end{array}}
$$

$$
\begin{array}{c|c}
 & \begin{array}{cc}\text{Sun} & \text{Rain}\end{array} \\
\begin{array}{c}\text{Sun} \\ \text{Rain}\end{array} & \begin{pmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{pmatrix}
\end{array}
$$

Sunny today

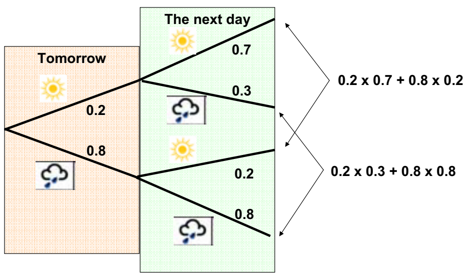


Sunny today

$$P(\text{Sunny on Sat}|\text{Sunny on Thurs}) = (0.7 \quad 0.2)\begin{pmatrix}0.7\\0.3\end{pmatrix} = 0.7 \times 0.7 + 0.3 \times 0.2$$

$$P(\text{Rain on Sat}|\text{Sunny on Thurs}) = (0.3 \quad 0.8)\begin{pmatrix}0.7\\0.3\end{pmatrix} = 0.7 \times 0.3 + 0.3 \times 0.8$$

Rainy today



Rain today 

P(Sunny on Sat|Rain on Thurs) = $\begin{pmatrix} 0.7 & 0.2 \end{pmatrix} \begin{pmatrix} 0.2 \\ 0.8 \end{pmatrix} =$ 0.2 x 0.7 + 0.8 x 0.2

P(Rain on Sat|Rain on Thurs) = $\begin{pmatrix} 0.3 & 0.8 \end{pmatrix} \begin{pmatrix} 0.2 \\ 0.8 \end{pmatrix} =$ 0.2 x 0.3 + 0.8 x 0.8

Matrix Multiplication

$$\begin{array}{cc} & \textbf{Thursday} \\ \textbf{Saturday} & \begin{pmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{pmatrix}^2 \end{array}$$

Matrix Multiplication

$$\begin{array}{cc} & \textbf{Thursday} \\ \textbf{Sunday} & \begin{pmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{pmatrix}^3 \end{array}$$

Matrix Multiplication

$$\begin{array}{cc} & \textbf{Thursday} \\ \textbf{Monday} & \begin{pmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{pmatrix}^4 \end{array}$$

What are these matrices?

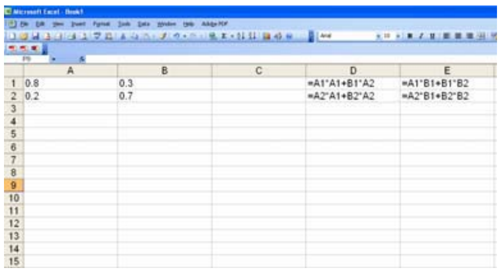
$$A^2 = \begin{pmatrix} 0.55 & 0.3 \\ 0.45 & 0.7 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 0.475 & 0.35 \\ 0.525 & 0.65 \end{pmatrix}$$

$$A^4 = \begin{pmatrix} 0.4375 & 0.375 \\ 0.5625 & 0.625 \end{pmatrix}$$

$$A^5 = \begin{pmatrix} 0.41875 & 0.3875 \\ 0.58125 & 0.6125 \end{pmatrix}$$

How could we use Excel to calculate high powers of a matrix quickly?



Modelling a situation with a Markov Chain

- Modelling is a skill that can be very weak in students.
- Markov chains is a good way to introduce modelling because the mathematics isn't too difficult.

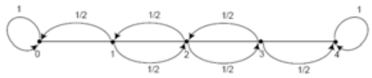
Example - Modelling

- We have two urns that, between them, contain four balls.
- At each step, one of the four balls is chosen at random and moved from the urn that it is in into the other urn.
- We choose, as states, the number of balls in the first urn.
- What is the transition matrix?

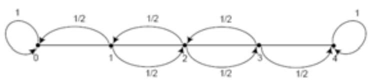
Tennis

- Consider the game of tennis at Wimbledon in the final set at 6-6.
- When A is serving she has a 0.99 chance of winning the game. When B is serving she has a 0.99 chance of winning the game
- Set this up as a Markov chain with.

Drunkard's Walk

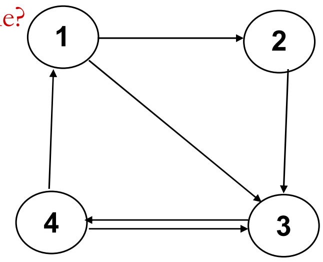


Drunkard's Walk



$$\begin{pmatrix} 1 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 1 \end{pmatrix}$$

Can you pagerank like Google?



Equilibrium Probabilities

$$A^{10} = \begin{pmatrix} 0.4005859375 & 0.399609375 \\ 0.5994140625 & 0.600390625 \end{pmatrix}$$

$$A^{100} = \begin{pmatrix} 0.4 & 0.4 \\ 0.6 & 0.6 \end{pmatrix}$$

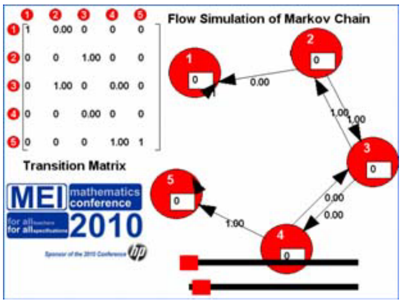
$$A^{1000} = \begin{pmatrix} 0.4 & 0.4 \\ 0.6 & 0.6 \end{pmatrix}$$

$$A^{10000} = \begin{pmatrix} 0.4 & 0.4 \\ 0.6 & 0.6 \end{pmatrix}$$

Ways to find equilibrium probabilities

- High powers of the matrix
- Fixed Point iteration
- Simulation
- Algebra

Simulating a Markov Chain



Use of dice, connections with Simulation chapter of D1

- Students should use dice/coins/random number generators to determine path through the system an instance.
- You should decide how many iterations an instance should live for until its final position is recorded.
- You'll need to think about initial conditions.

Other things to investigate

- Run length
- Absorbing States
- Reflective Barriers
- Regular Chains
- Ergodic Chains
- Periodic Chains

Run length

- Give a Markov Chain what is the expected number of times it will stay in a particular state?

Back to…

- **An island**
  If it rains today the probability that it will rain tomorrow is 0.8 and the probability that it will be sunny is 0.2.

Today is Thursday and its raining. What the probability that is will rain until Saturday and then be sunny on Sunday?

$$0.8\times0.8\times0.2$$

Expected run length

The expected run length for rainy days would be

$$(0\times0.2)+(1\times0.8\times0.2)+(2\times0.8^2\times0.2)+(3\times0.8^3\times0.2)+\dots$$

$$=[0.8\times0.2](1+2\times0.8+3\times0.8^2+\dots)$$

$$=(0.8\times0.2)\times\frac{1}{(1-0.8)^2}$$

Expected run length

- In general the expected run length is

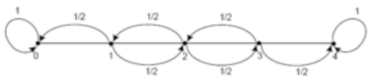
$$E(X)=\alpha(1-\alpha)\times\frac{1}{(1-\alpha)^2}=\frac{\alpha}{1-\alpha}$$

- There are lots of questions you might ask students about this (about the effects of changing α)

Absorbing State

- If a Markov chain has an absorbing state then eventually the system will go into one of the absorbing states.

Absorbing State Example - Drunkard's Walk



- Absorbing States – a state you can't leave
- Transient States – all other states

Reflecting Barrier

- When the probability of changing from state i to state j is 1, state i is said to be a reflecting barrier.
- Questions for students: what would the transition matrix look like when there is a reflecting barrier?

Periodic chain

- A periodic chain is one for which the is a value of k such that $P^k = P$, where P is the transition matrix of the chain.
- The period of such a chain is then said to be k – 1 where k is the smallest such value.

More terminology

- A Markov chain is called an ergodic chain if it is possible to go from every state to every state
- A Markov chain is called regular if some power of the transition matrix has only strictly positive entries

Questions

- Is an ergodic chain necessarily regular?
- Is an regular chain necessarily ergodic?

Answers

- A regular chain must be ergodic
  The fact that a power of the transition matrix has entirely positive entries shows that it must be possible to go from any state to any other state.
- Ergodic does not imply regular
  Can you think of a counter example?