Pictures and puzzles in teaching A level Core

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In this session we will look at a selection of puzzles which depend on ideas from AS and A2 Core Maths and which are accessible to A level students of a wide range of abilities. Some are designed to help students understand mathematical concepts through pictures, others to practise techniques with answers that are meaningful to the students; they should be able to decide for themselves if they've got them right.

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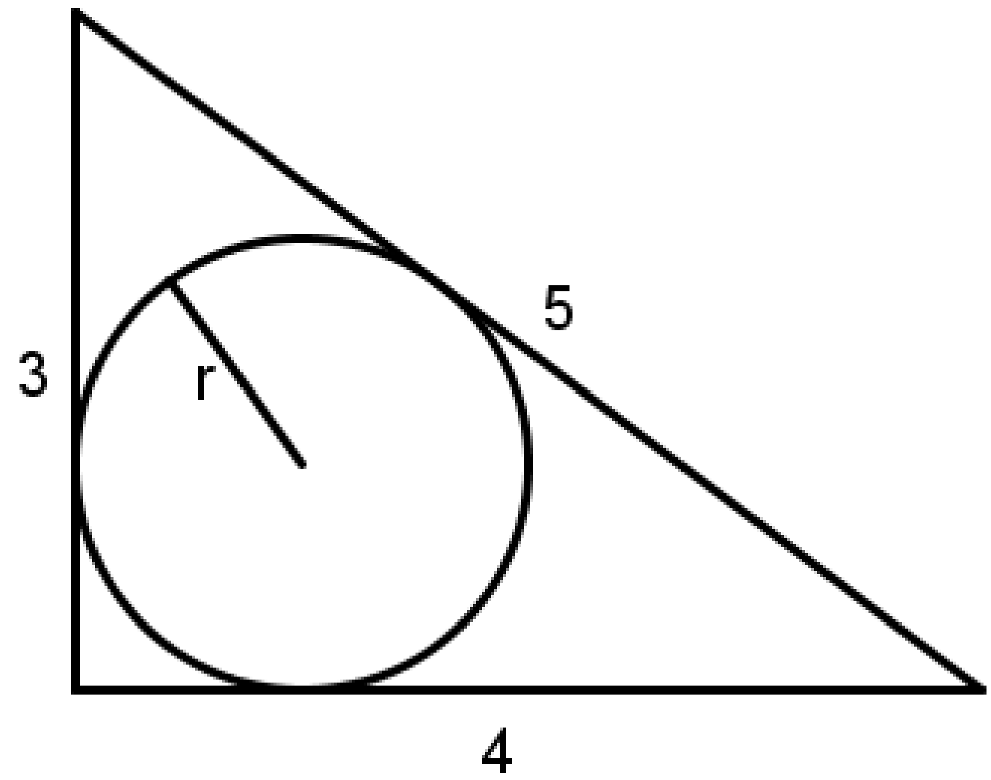
What is the radius of the biggest circle that can fit inside a 3-4-5 triangle?



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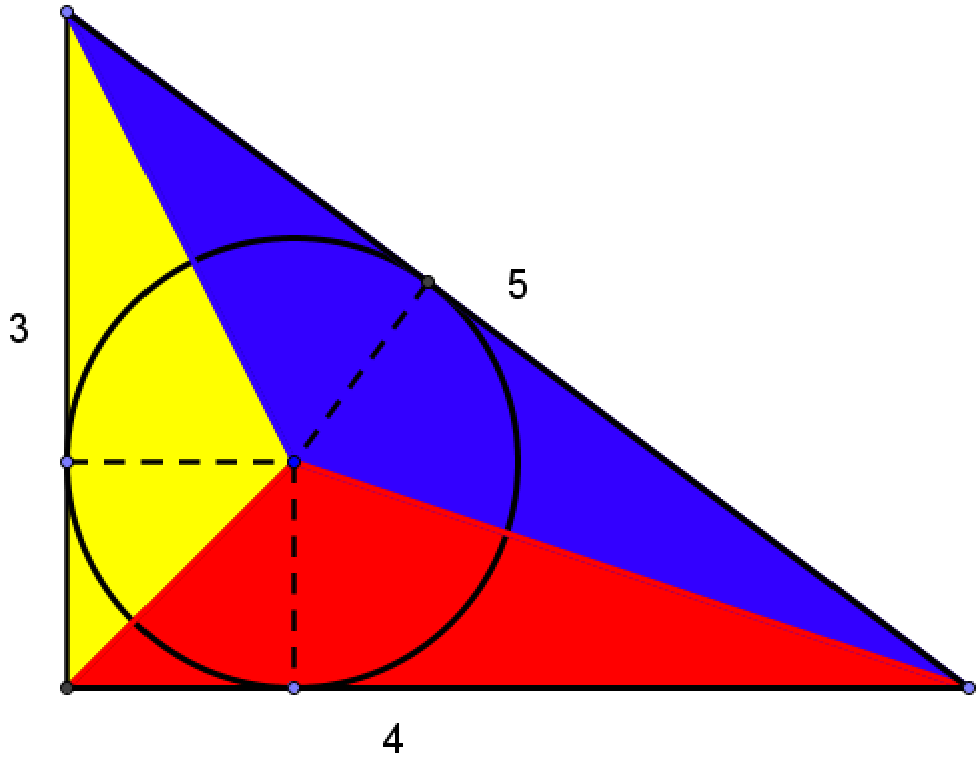

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1. Deepening understanding of algebra by building on familiar number and shape concepts

2. Numerical concepts in context: surds, indices, logs, sequences, combinations

3. Pythagoras: number & algebra & shape

4. Motivating proof through diagrams

5. Calculus

6. Resources which allow self checking through ICT, sketching, matching, …

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1. Deepening understanding of algebra by building on familiar number and shape concepts

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Squaring numbers ending in a 5

$$35^2 = 1225$$

$$3 \times 4 \qquad 5^2$$

Does this always work?

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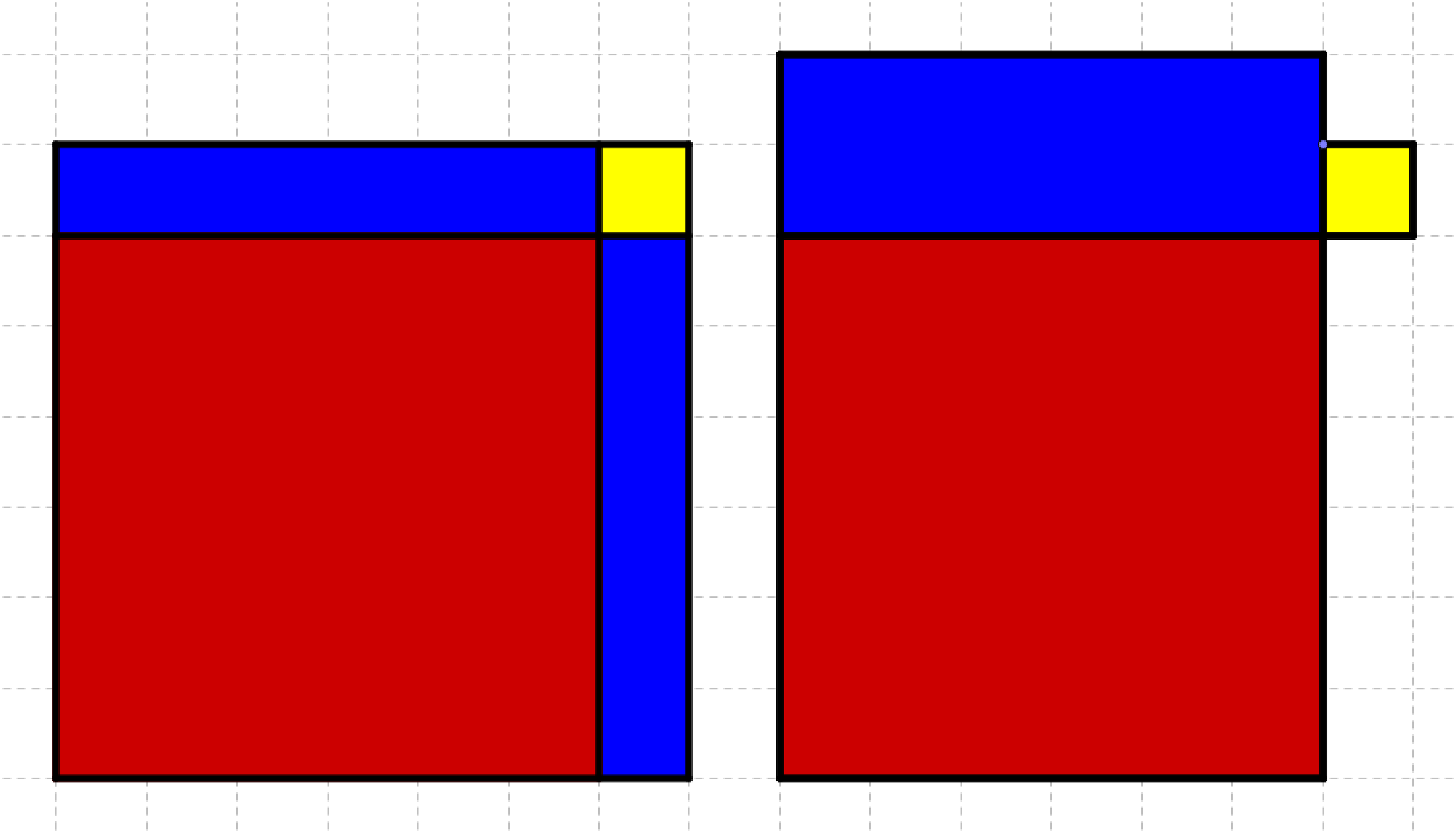

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6

3 8

1 5

$$68^2+85^2+51^2+13^2+36^2=15915$$

$$63^2+31^2+15^2+58^2+86^2=15915$$

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Add two square numbers together and double the answer.

Is it possible to find two square numbers which add to give this answer?

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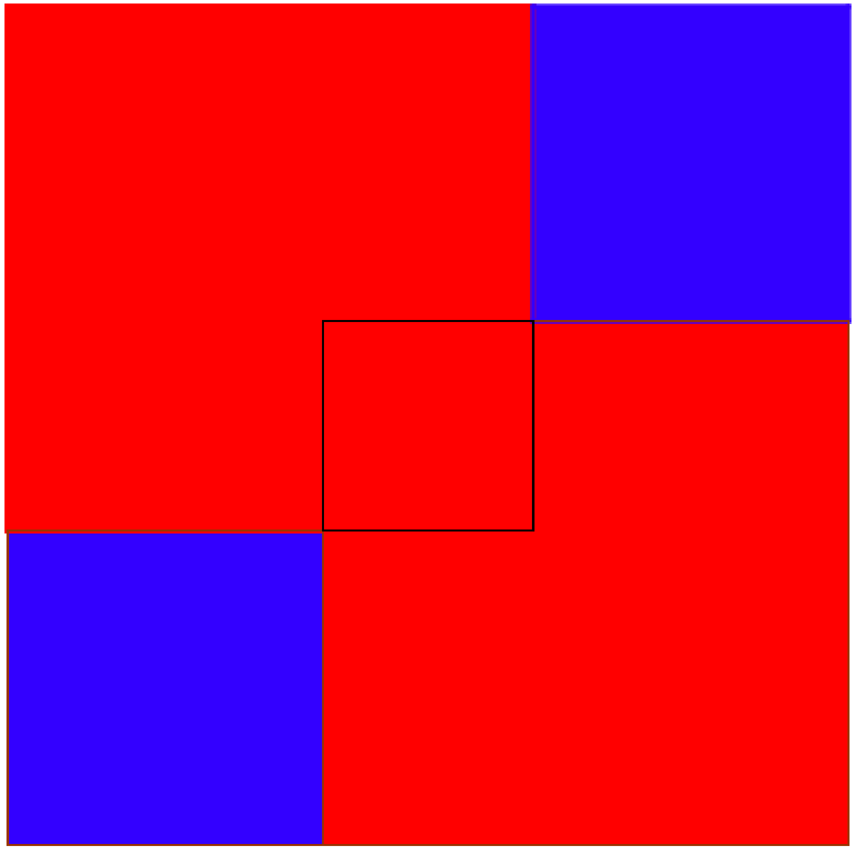

$$2\left(x^2+y^2\right)=\left(x+y\right)^2+\left(x-y\right)^2$$

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4 is an interesting square number because it is one greater than a prime number.

Find another square with this property.

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Visualising a binomial expansion

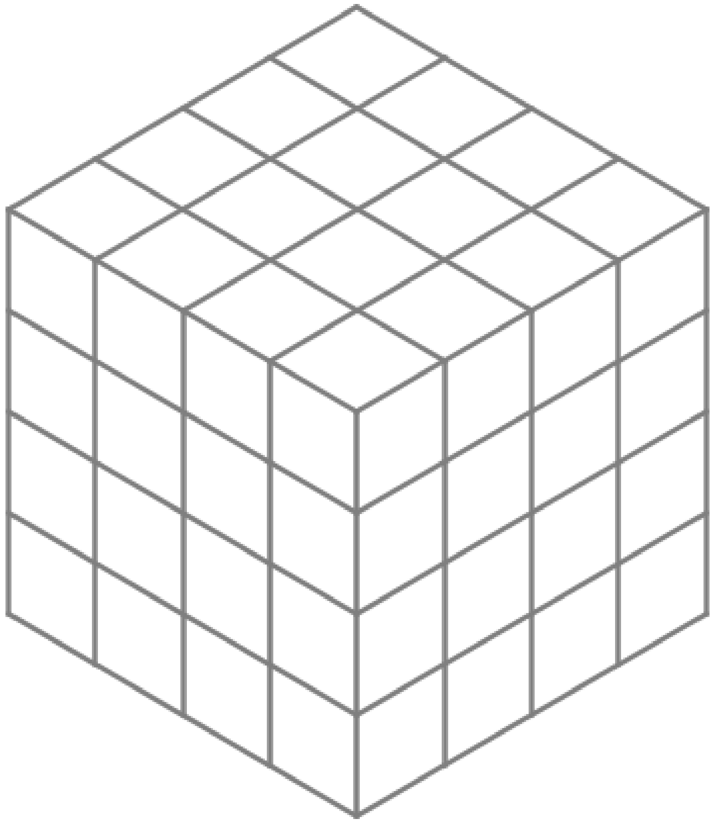
Imagine a cube of side length x and surrounding it by another layer of cubes.

First cover each face…

…then fill in the edges…

…then fill the gaps at the corners.

$$\left(x+2\right)^{3} \equiv$$



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2. Numerical concepts in context: surds, indices, logs, sequences, combinations

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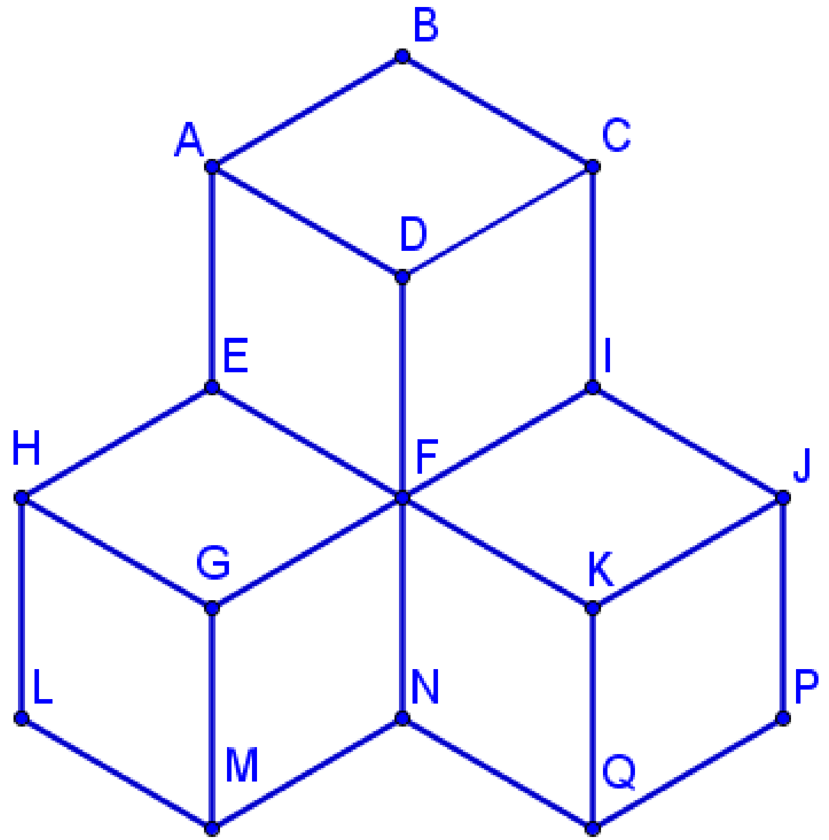
Four cubes



How many different lengths are there between pairs of vertices?

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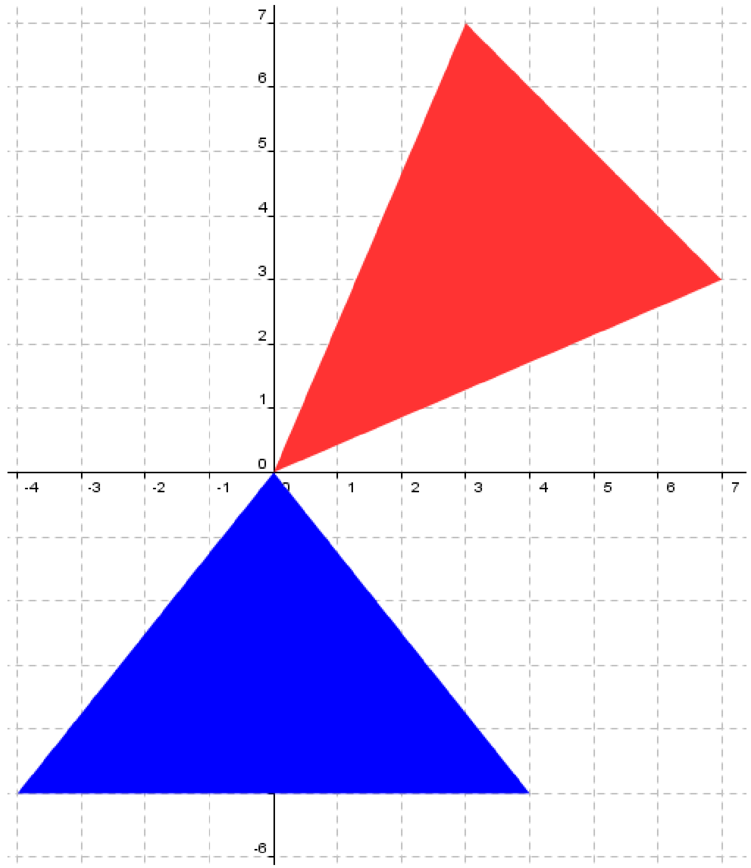
Isosceles triangles, equal sides meet at the origin, vertices on grid points, area 20 square units.



How many more can you find?

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$$
\begin{aligned}
3^2 > 2^3 \quad &\Rightarrow \quad \log_2 3^2 > \log_2 2^3 \\
&\Rightarrow \quad 2\log_2 3 > 3 \\
&\Rightarrow \quad \log_2 3 > 1.5
\end{aligned}
$$

What are the best upper and lower bounds you can find for $\log_2 3$?

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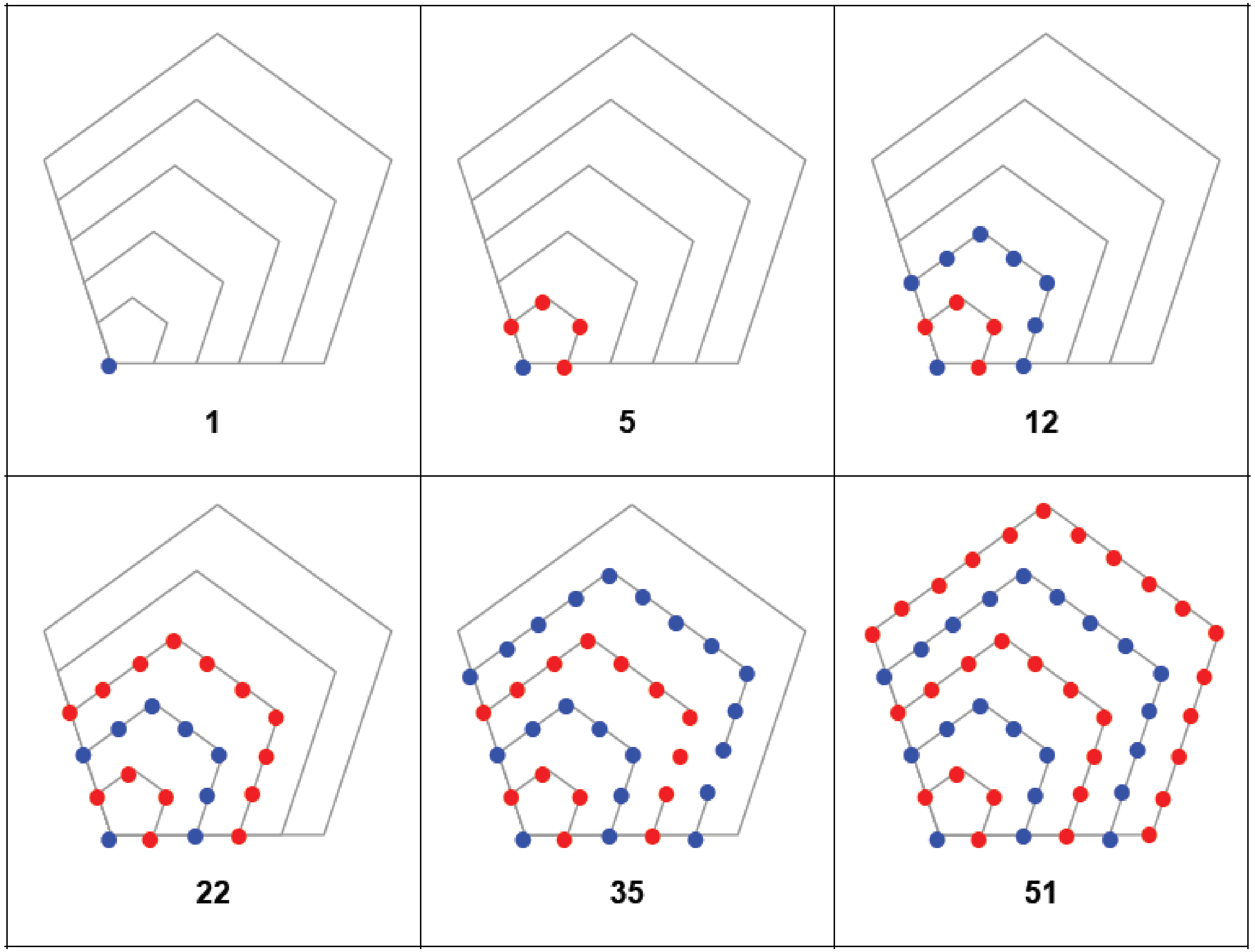
Pentagonal Numbers



What is the 100th pentagonal number?

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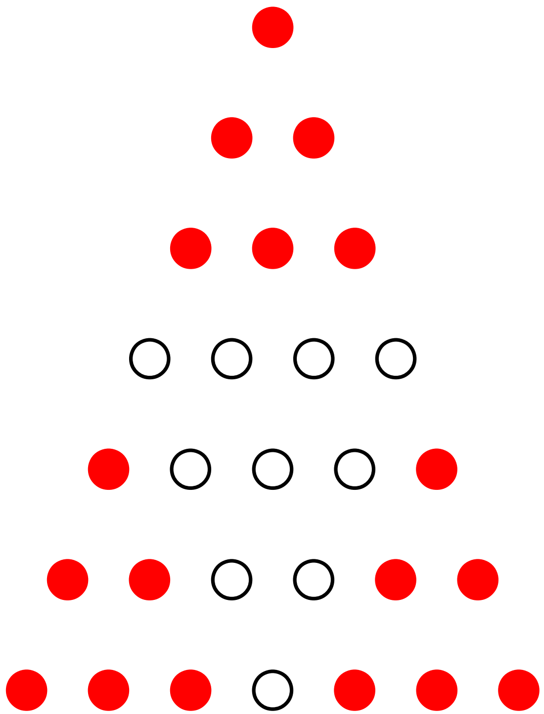

This diagram gives insight into why

$$3\Delta_n + \Delta_{n+1} = \Delta_{2n+1}$$

You could prove this algebraically

$$\frac{3n(n+1)}{2} + \frac{(n+1)(n+2)}{2} \equiv \frac{(2n+1)(2n+2)}{2}$$

- Can you find a similar relationship for Δ_{2n} rather than Δ_{2n+1}?
- Can you find any other relationships between triangular numbers or between triangular numbers and square numbers?

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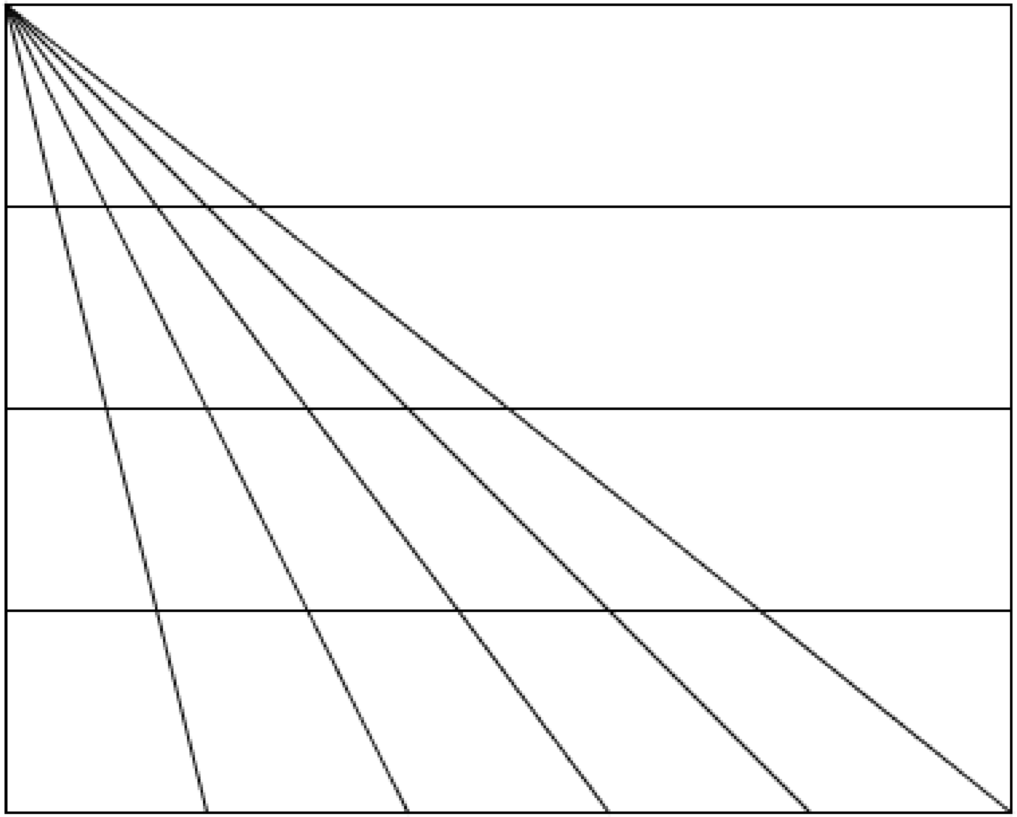

How many triangles can you see?

How many quadrilaterals?

How many trapezia?

What if there were r rows rather than 4, and the bottom edge was divided into s sections rather than 5?

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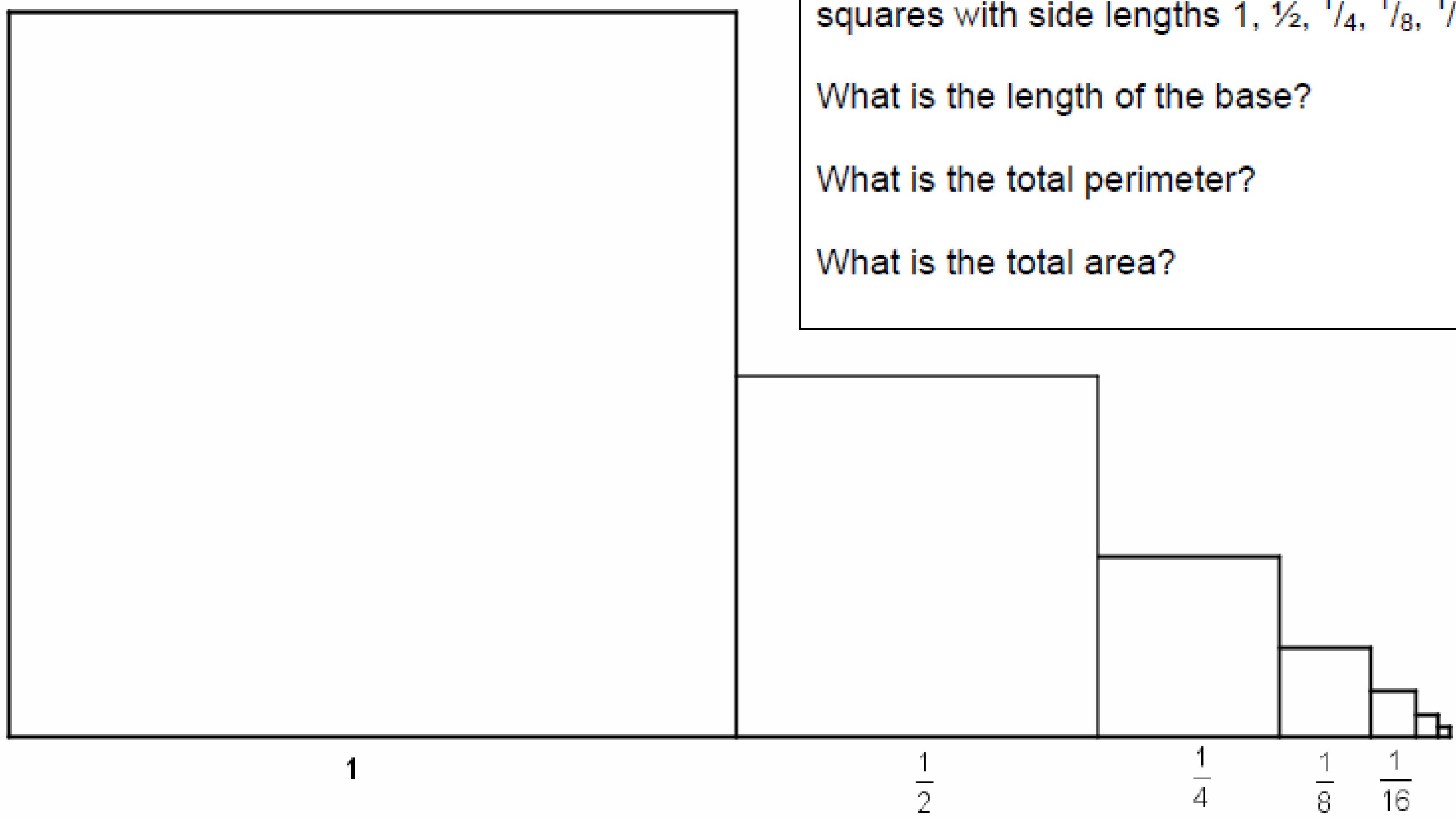
The diagram shows an infinite number of squares with side lengths 1, ½, $^1/_4$, $^1/_8$, $^1/_{16}$, . . .

What is the length of the base?

What is the total perimeter?

What is the total area?



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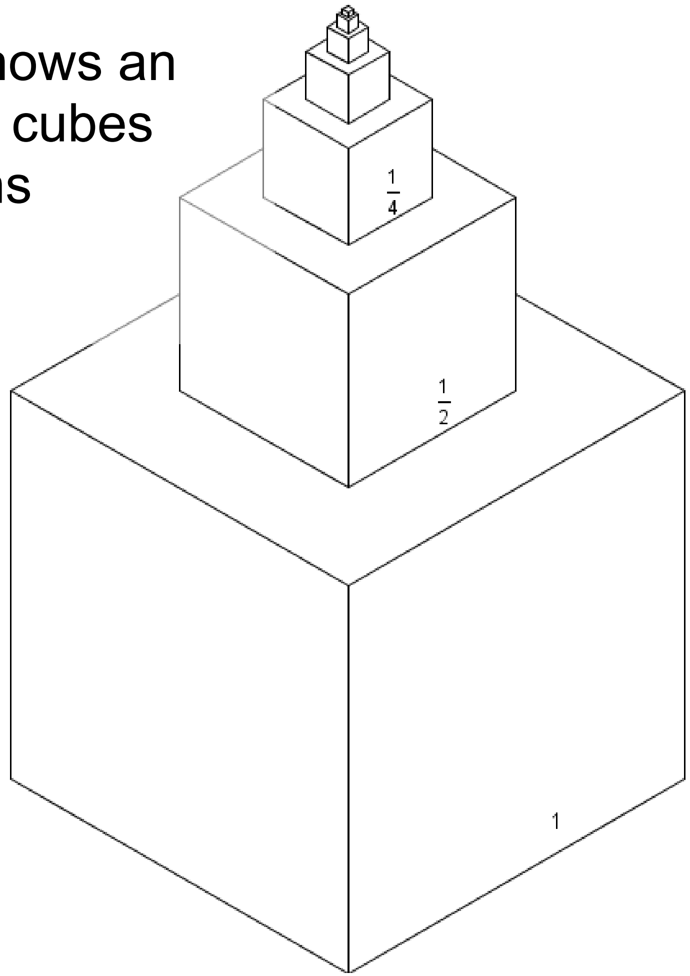
The diagram shows an infinite tower of cubes with side lengths

$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$



How high is the tower?

What is the exposed surface area?

What is the volume of the object?

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3. Pythagoras:

number & algebra & shape

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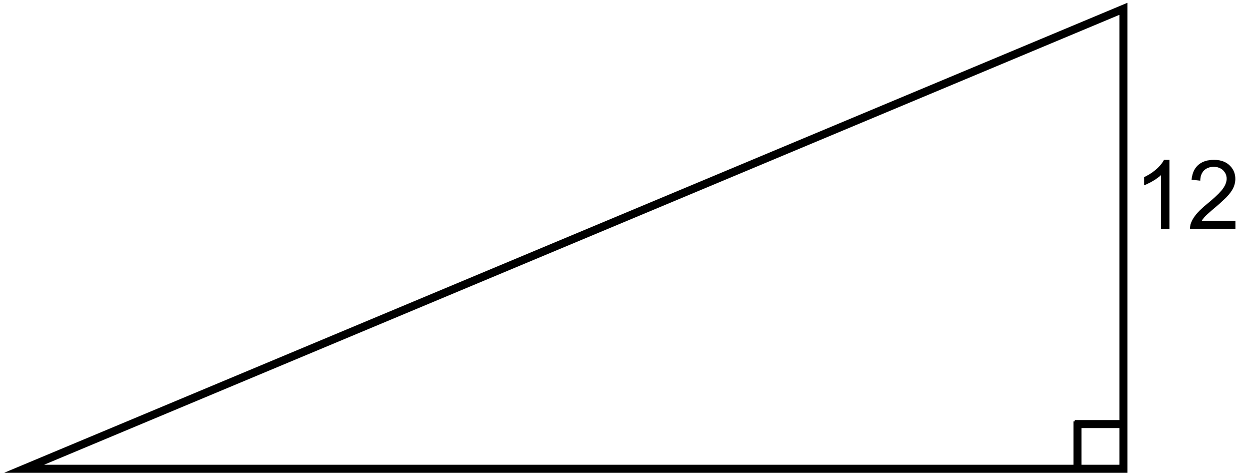
How many right-angled triangles can you find with integer length sides where the shortest side is 12 units?



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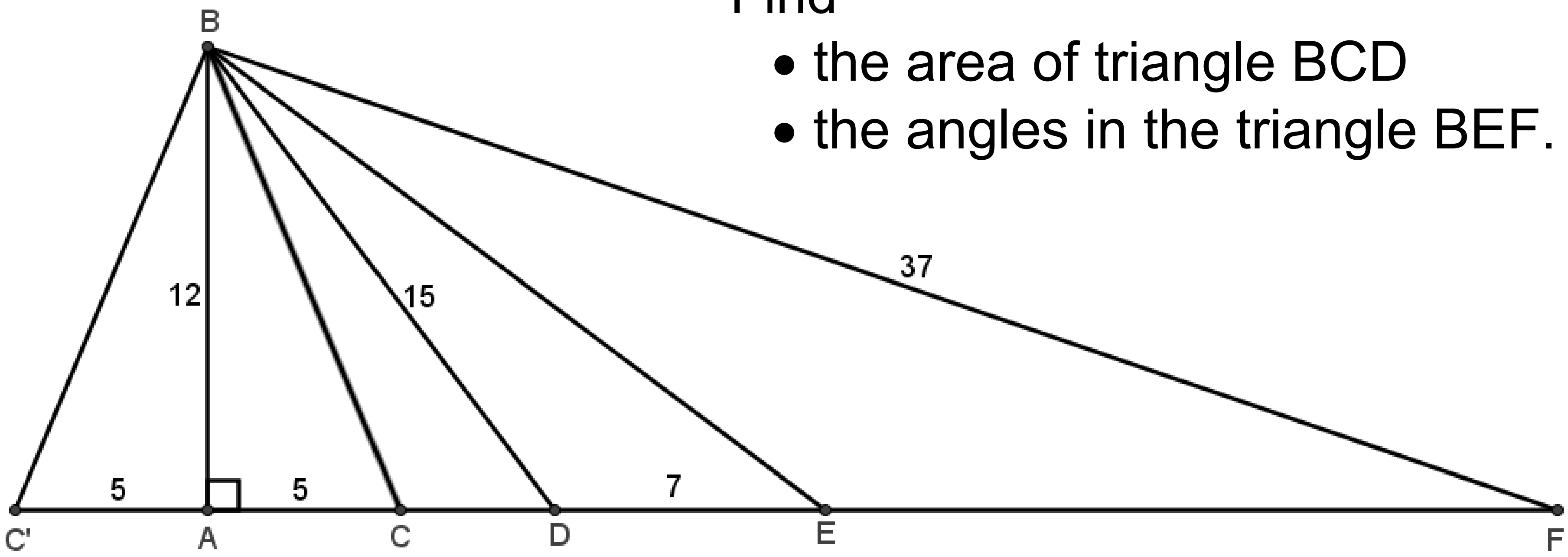
C′ACDEF is a straight line.

Find

- the area of triangle BCD
- the angles in the triangle BEF.



Using information from this diagram, make up **non-calculator** questions involving

(i) double angle formulae, and (ii) compound angle formulae

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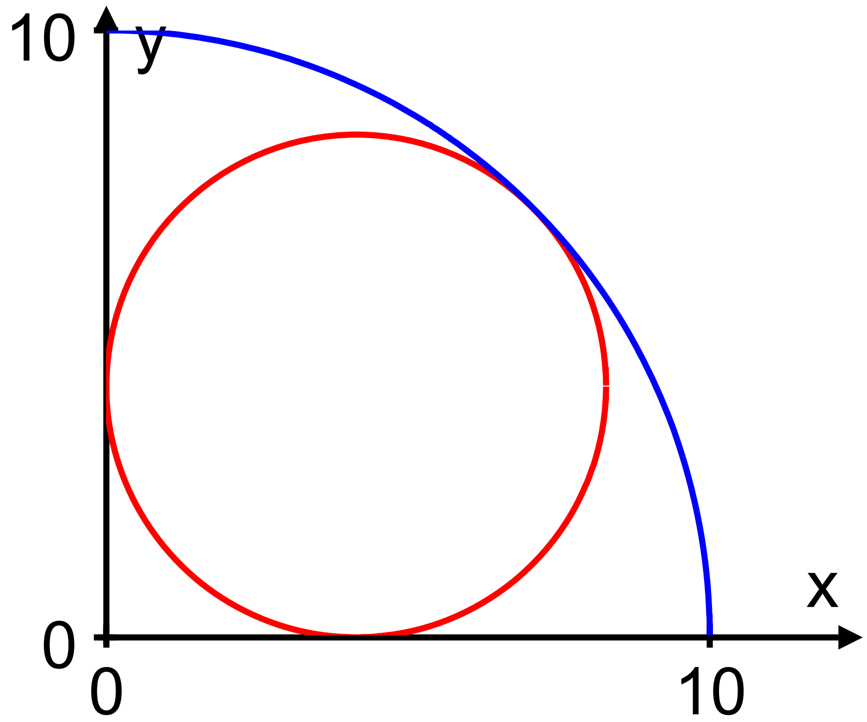

What is the **exact** radius of the small circle?

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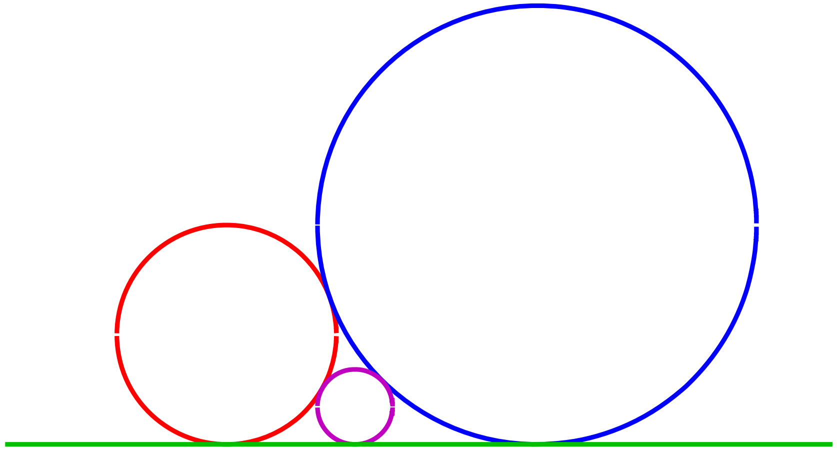
Common tangent

The three circles shown below touch each other and all three have a common tangent. If the larger circles have radii 1 and 2, find the radius of the smallest circle.



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4. Motivating proof through diagrams

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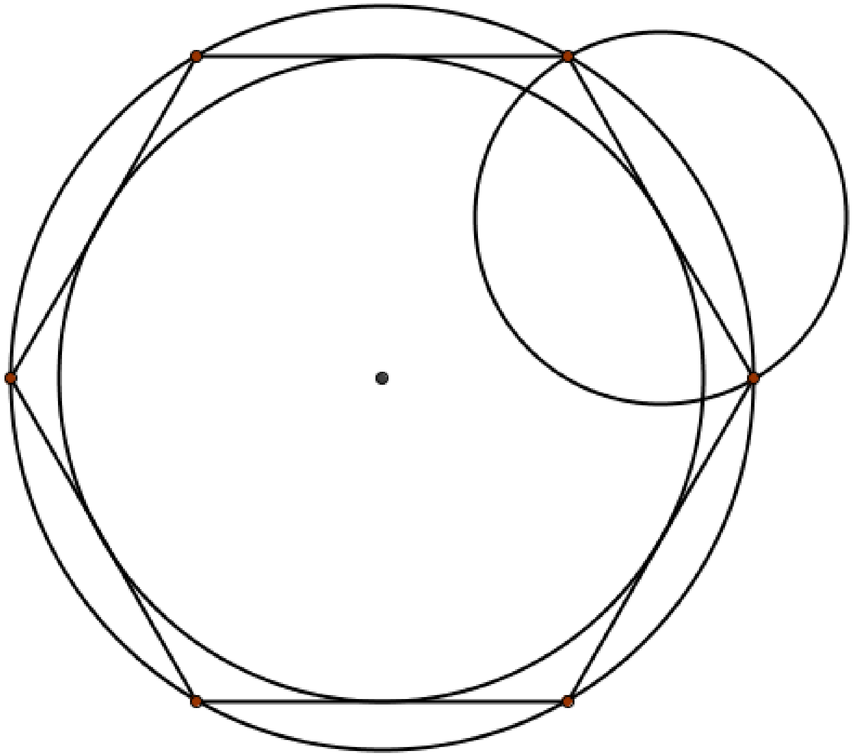

What is the connection between the areas of the three circles?

What about for other regular polygons?

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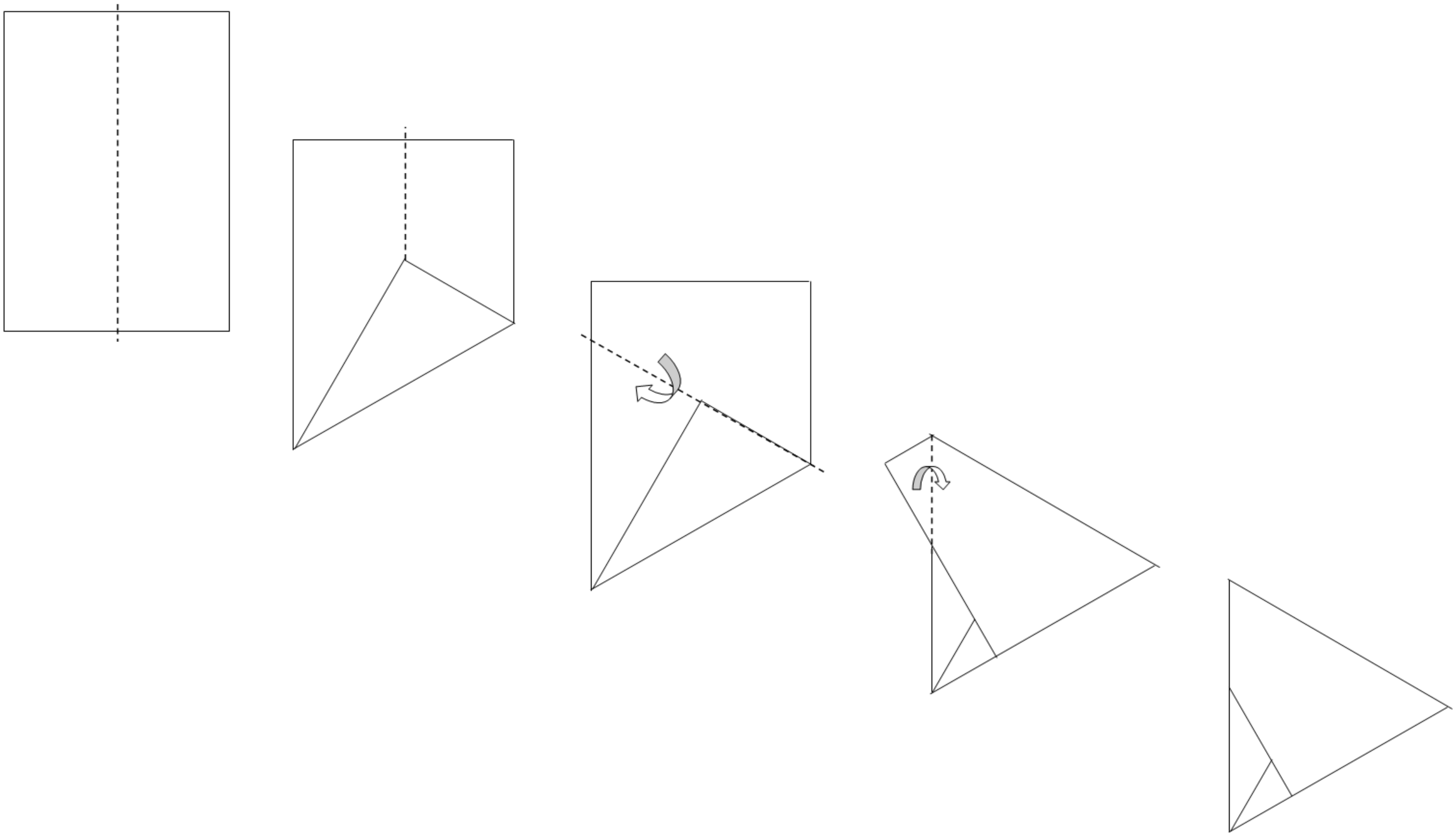
Equilateral?



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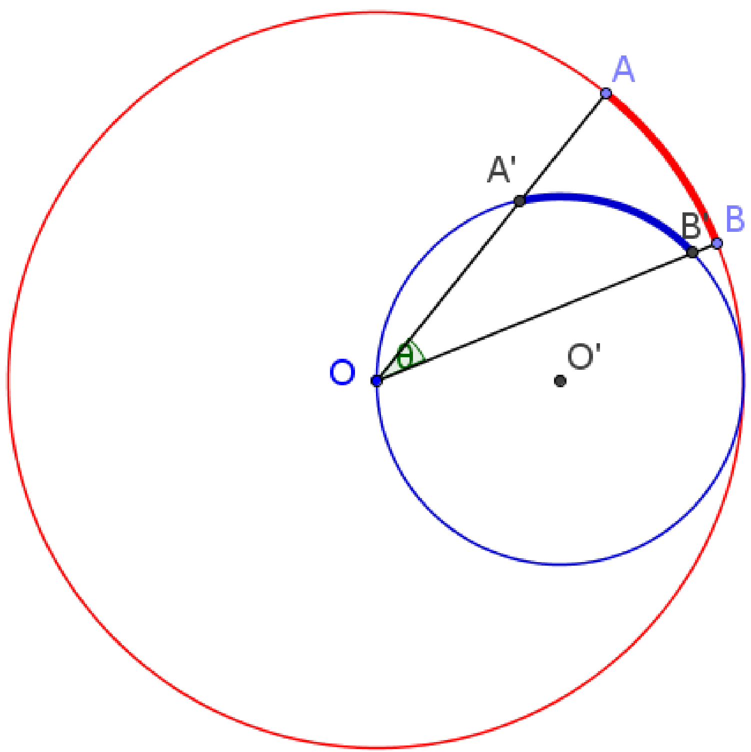
The diagram shows two circles, the radius of the larger one being the diameter of the smaller one.



Prove that the arc lengths AB and A’B’ are equal.

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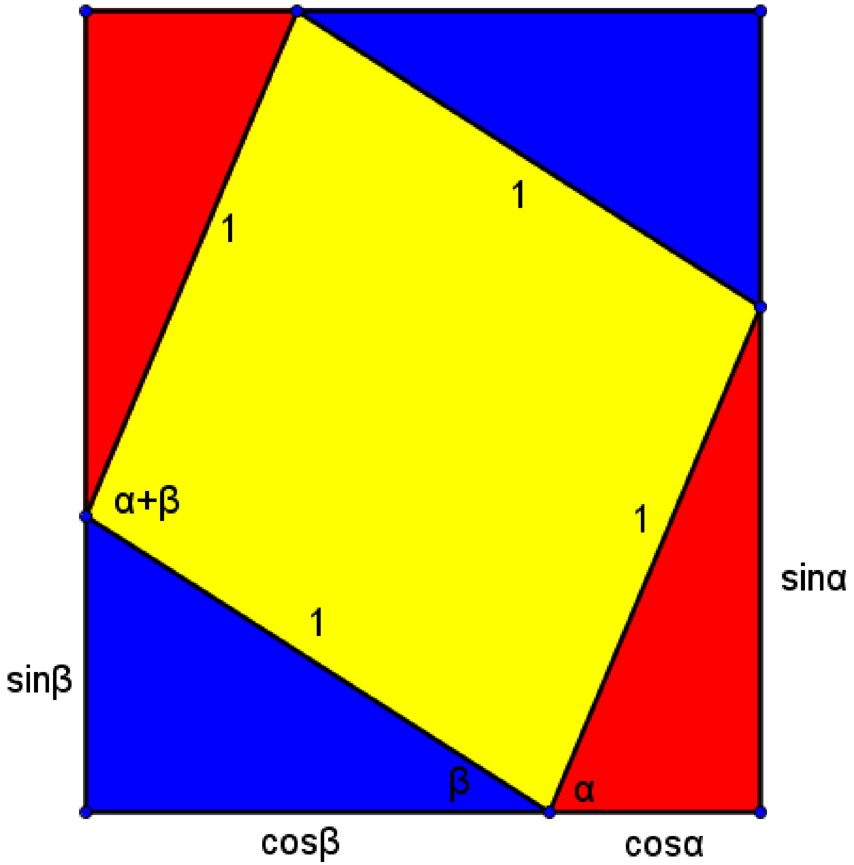
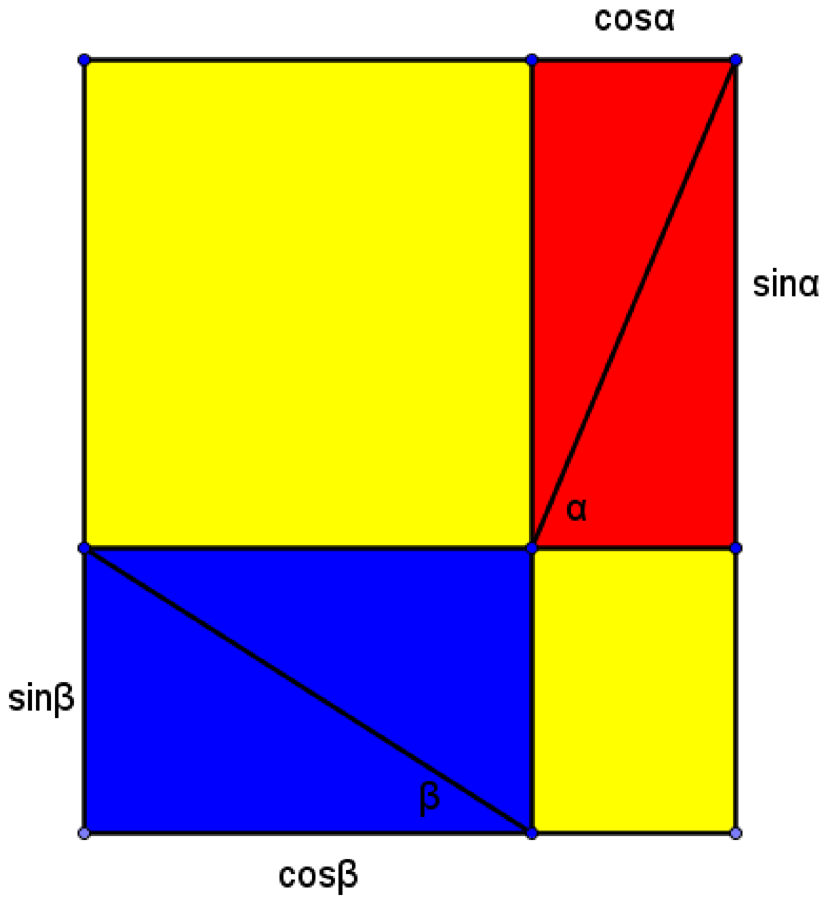
$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$





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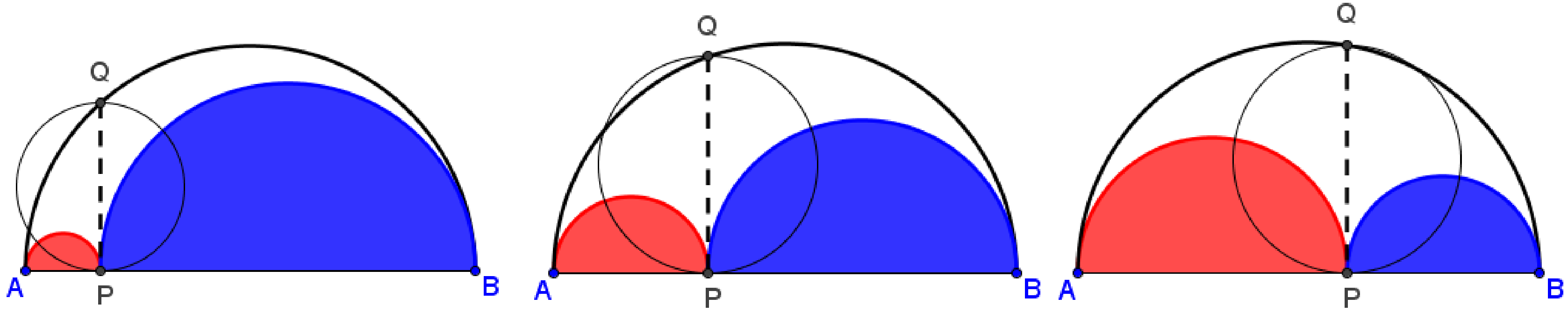
Arbelos ('Shoemaker's knife')

This problem dates back to Archimedes. The large semicircle has diameter AB. A point P is chosen on AB and the semicircles with diameters AP and PB are drawn.



Show that the area inside the large semicircle but outside the two smaller ones is equal to the area of the circle with diameter PQ, where Q lies on the large semicircle and PQ is perpendicular to AB.

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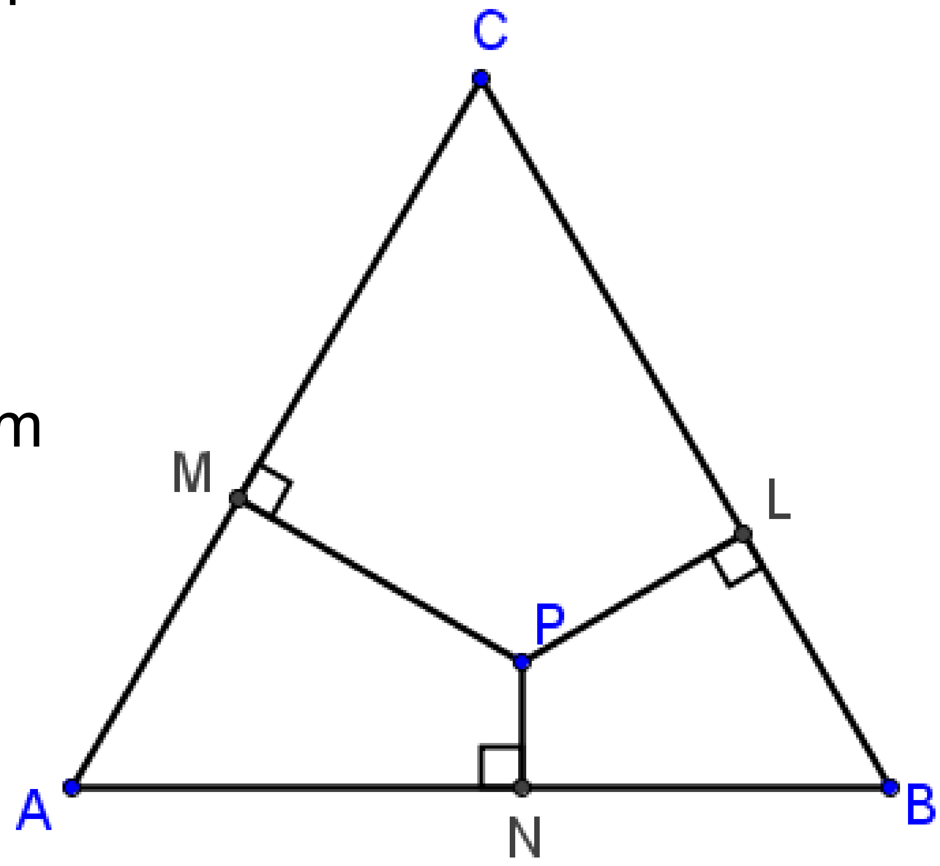
ABC is an equilateral triangle of side length 6cm.

P is a point inside the triangle.

Where should you place P to maximise the total distance from the three sides, PL+PM+PN?



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5. Calculus

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Some positive numbers add up to 19.

What is the maximum product?

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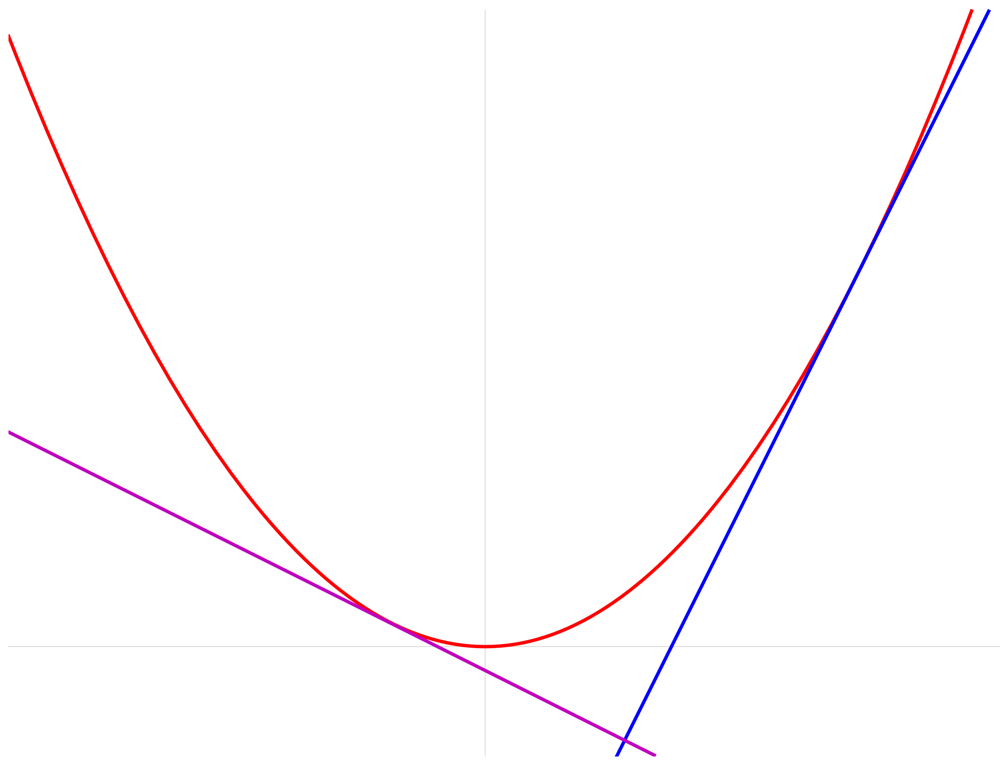
How would you find two **perpendicular** tangents to a curve?



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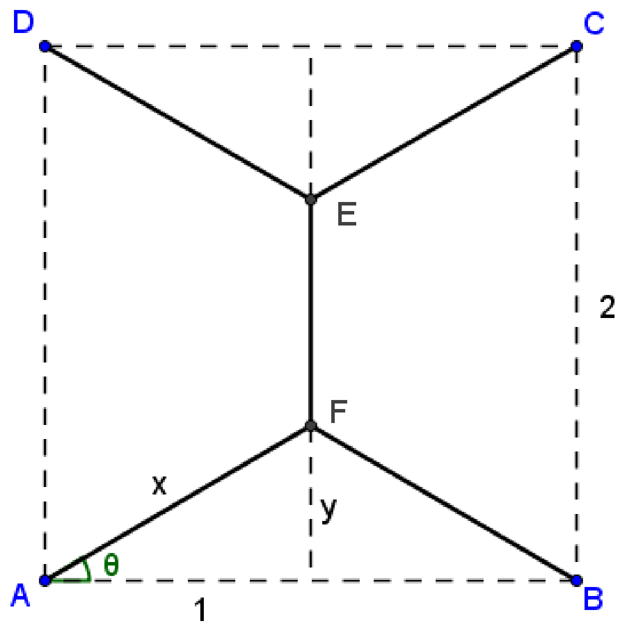
Four towns, A, B, C and D, are situated at the corners of a square of side length 2km.

They are to be connected by a road network so that the total length of the road system is minimised. One symmetrical system is given here.



What angle θ would minimise this particular road length?

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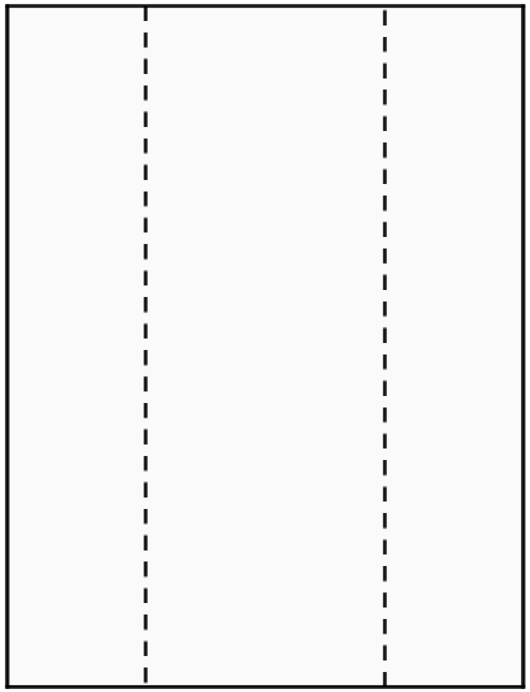
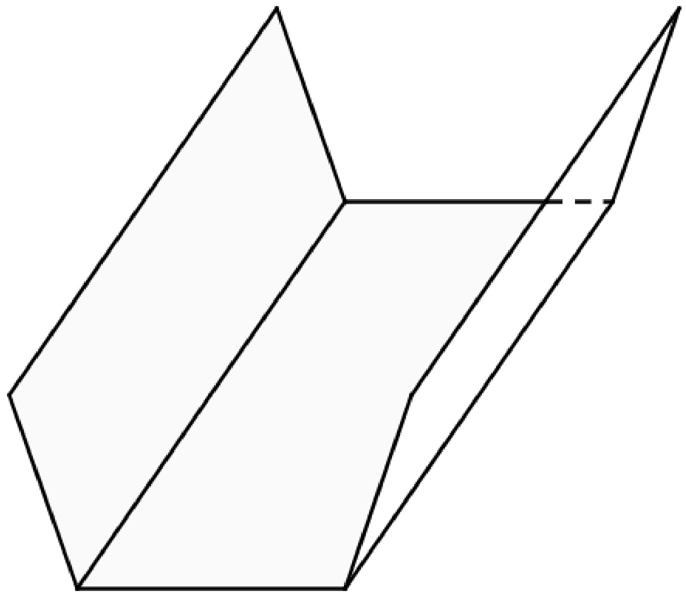
You have a rectangular sheet of metal. Two folds are made, both parallel to the longest edge, and the outer edges are folded inwards to create a gutter.





In order to maximise the cross sectional area of the gutter, where should you position the folds and at what angle should you then fold the outer edges up?

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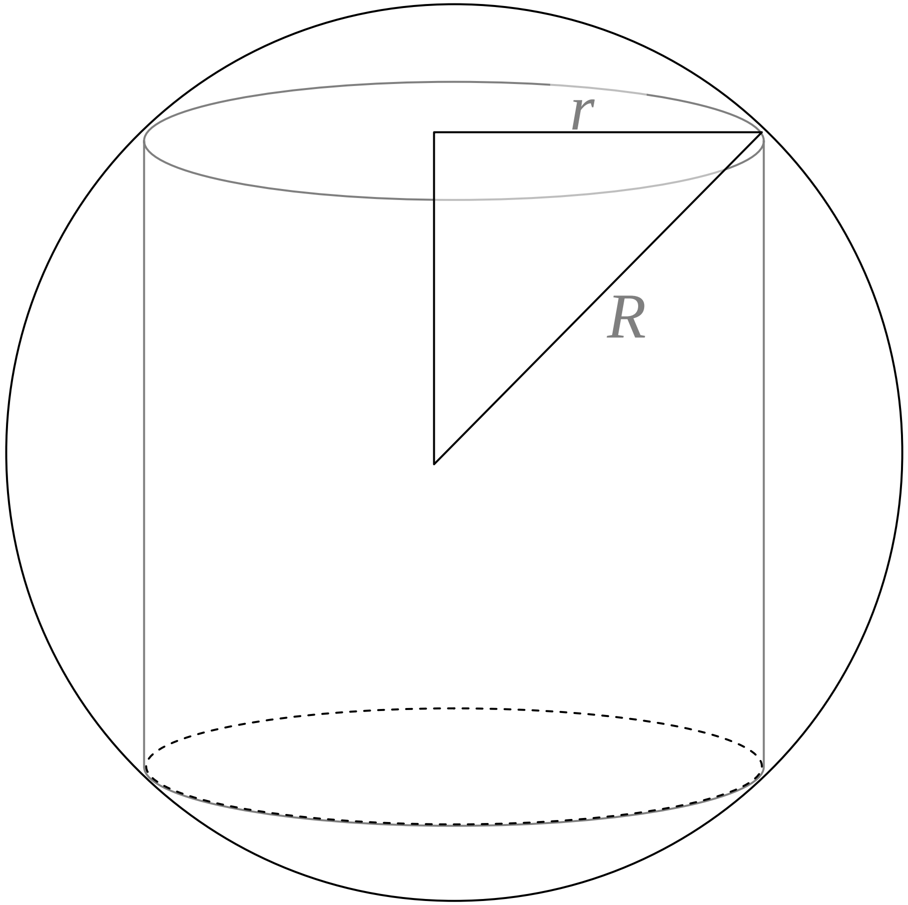
Largest cylinder

What is the largest cylinder by volume that can be inscribed in a sphere of radius R?



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6. Resources which allow self checking through ICT, sketching, matching, …

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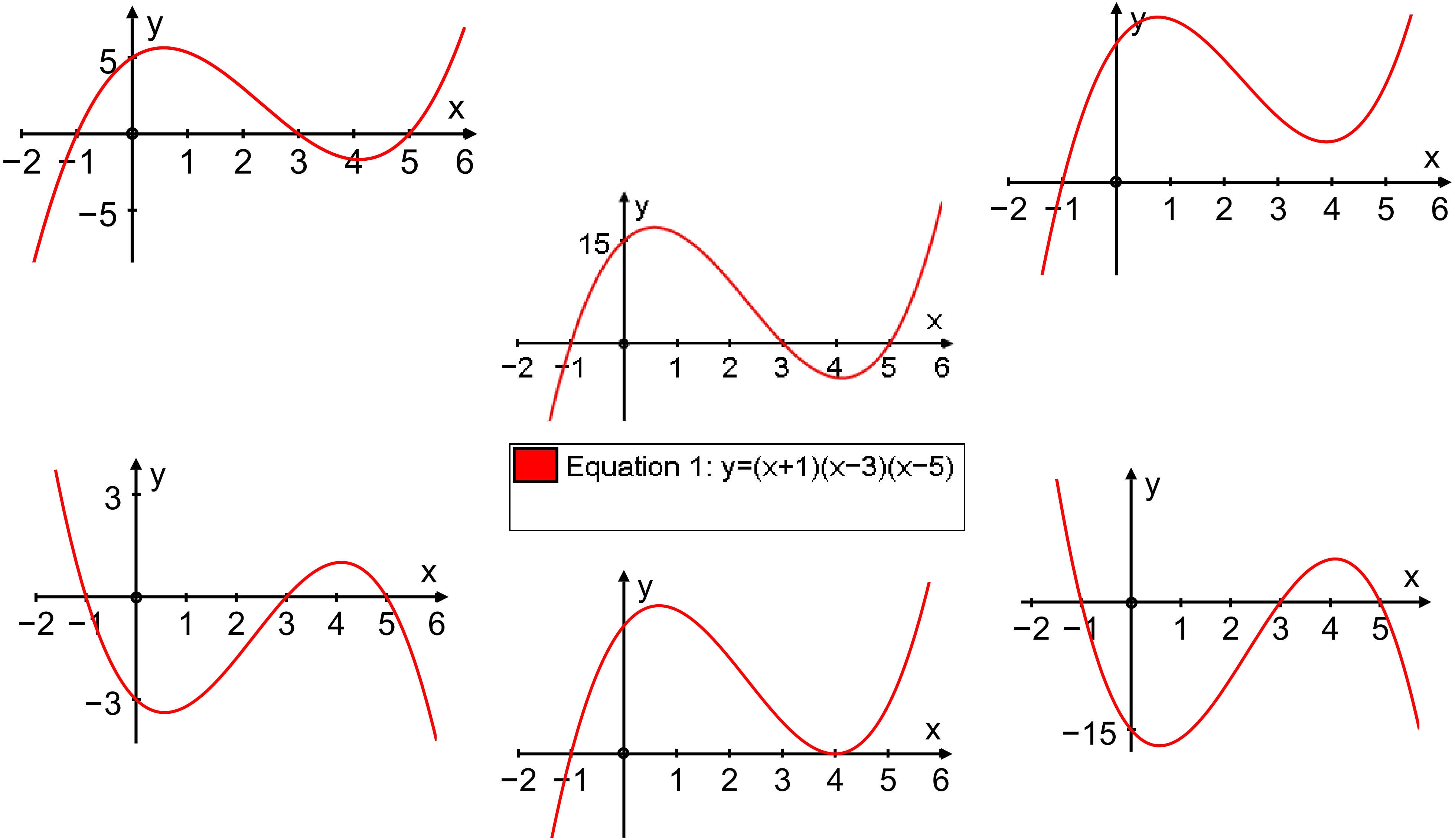

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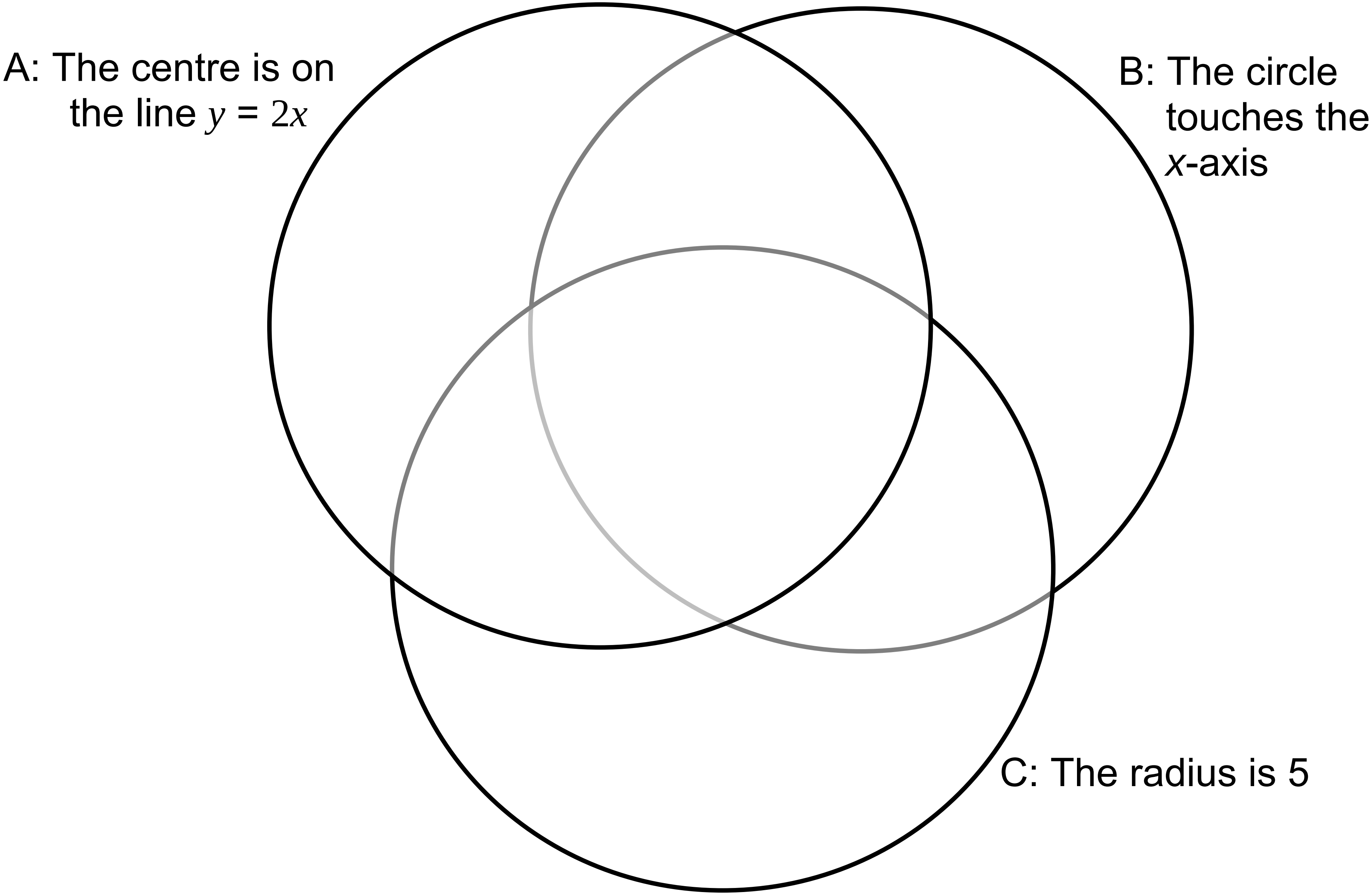
A: The centre is on the line $y = 2x$
B: The circle touches the x-axis
C: The radius is 5

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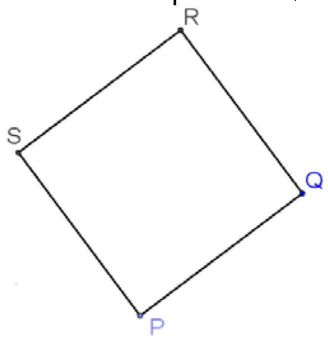
A. P:(0,1) and Q:(4,4) are corners of the square PQRS: 	B. The coordinates of R and S	C. The gradient of PR	D. The equation of PR
E. The area of the square	F. The length of PQ	G. The midpoint of PR	H. The area of the circle passing through the four corners of the square
I. The coordinates of both points where the square meets the *y*-axis	J. The length of QS	K. The equation of the perpendicular bisector of side QR	L. The coordinates of the point where the line QS meets the *x*-axis.

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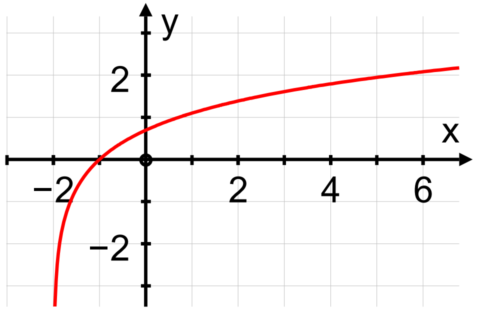
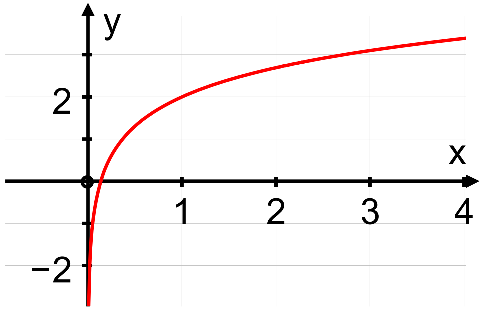
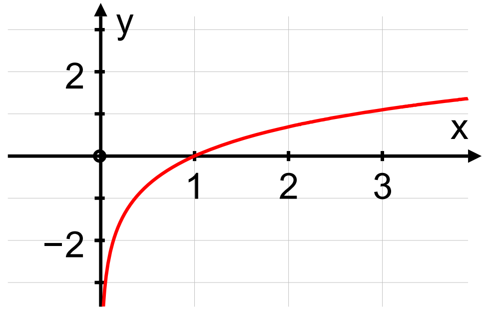


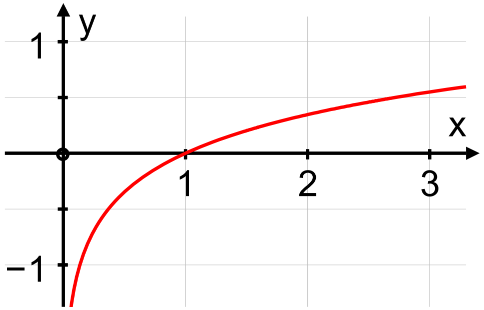
A



B



C



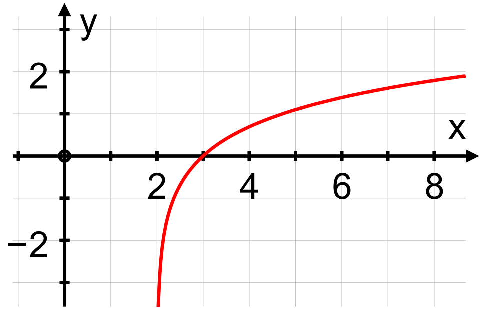
D

$y+2=\ln x$ $\qquad$ $y=\ln 2x$

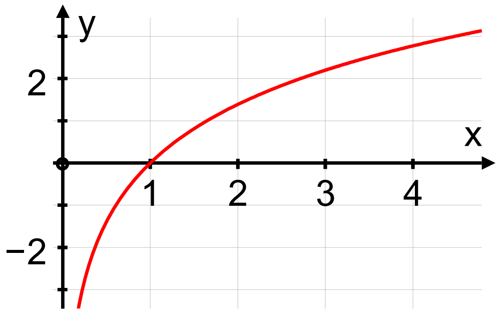
$y=2\ln x$ $\qquad$ $y=\ln(x+2)$

$y=\ln(x-2)$ $\qquad$ $y=\ln x$

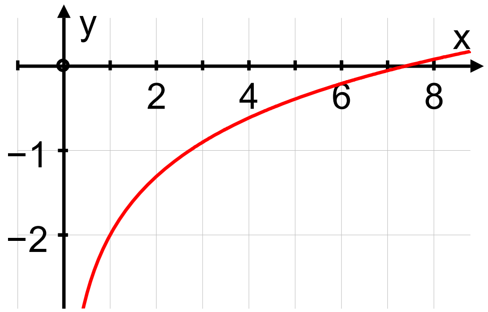
$y=2+\ln x$ $\qquad$ $2y=\ln x$



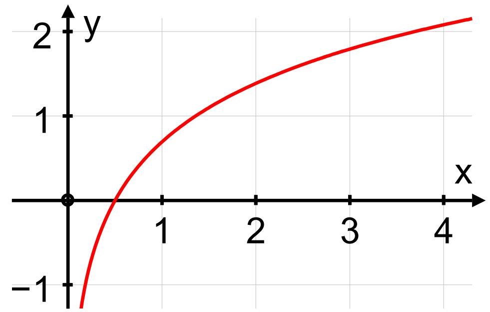
E



F



G



H

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Can you find a function for each domain-range combination?

Range \ Domain	$x \in \mathbb{R}$	$x \in \mathbb{R}, x > 0$
$y \in \mathbb{R}$		
$y \in \mathbb{R}, y > 1$		
$y \in \mathbb{R}, \lvert y \rvert \leq 1$		

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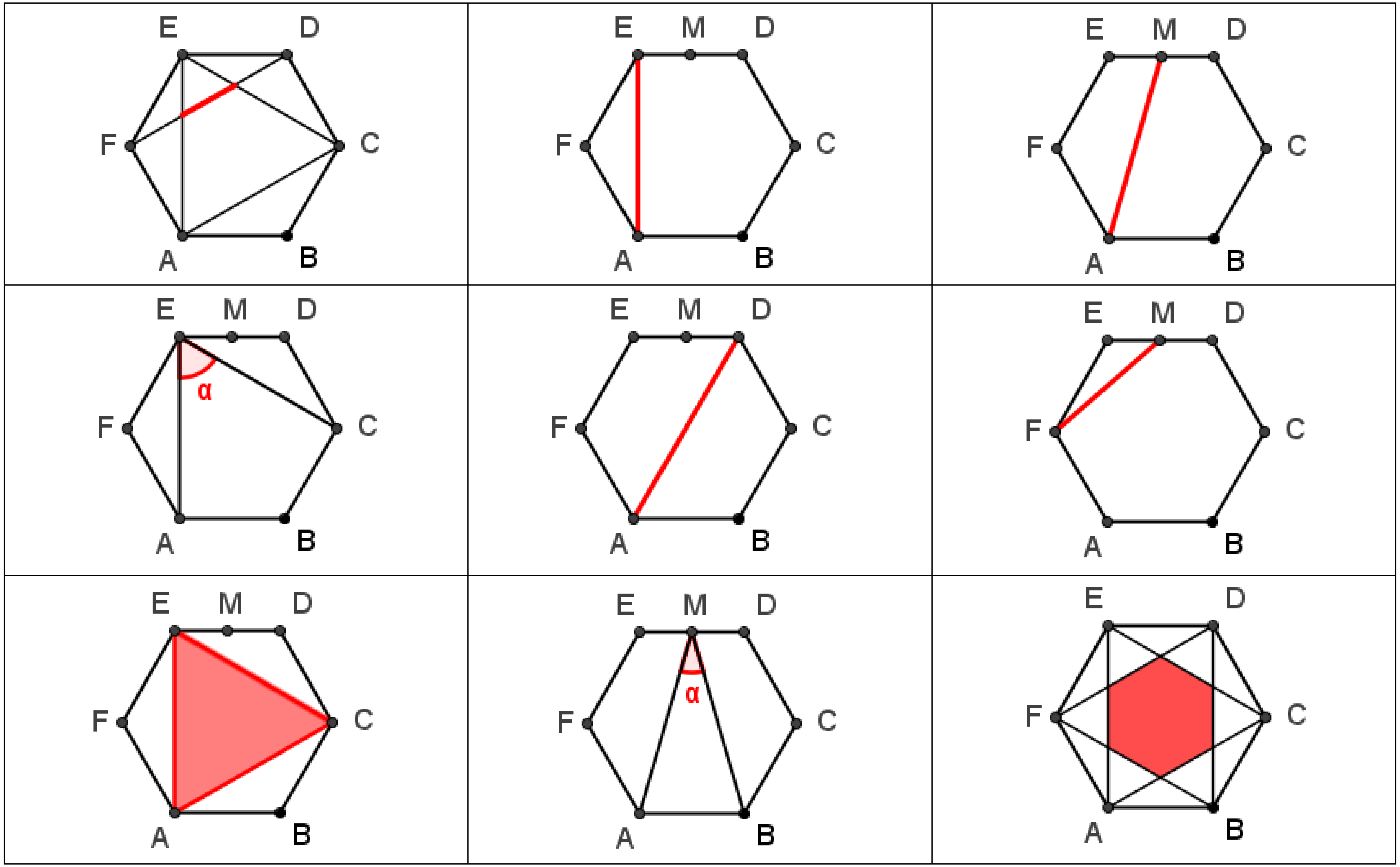

On each card ABCDEF is a regular hexagon of side length 2cm. M is the midpoint of ED.

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$t = 1$ gives the point $(1, 2)$	$x = 2t + \frac{1}{t}$	$y = \frac{1}{t^2}$	$\frac{dx}{dt} = 1$	$\frac{dy}{dt} = \frac{1}{t}$	At $t = 1$, $\frac{dy}{dx} = 2$
$t = 1$ gives the point $(1, 1)$	$x = 4t + t^2$	$y = \ln t$	$\frac{dx}{dt} = 2t$	$\frac{dy}{dt} = -1$	At $t = 1$, $\frac{dy}{dx} = \frac{3}{2}$
$t = 1$ gives the point $(5, 2)$	$x = t$	$y = t^3$	$\frac{dx}{dt} = \frac{2}{\sqrt{t}}$	$\frac{dy}{dt} = 3t^2$	At $t = 1$, $\frac{dy}{dx} = \frac{1}{2}$
$t = 1$ gives the point $(5, 1)$	$x = 4\sqrt{t}$	$y = t^2 + 1$	$\frac{dx}{dt} = 2 - \frac{1}{t^2}$	$\frac{dy}{dt} = 6t$	At $t = 1$, $\frac{dy}{dx} = -\frac{1}{6}$
$t = 1$ gives the point $(4, 0)$	$x = t^2$	$y = 3t^2$	$\frac{dx}{dt} = 3$	$\frac{dy}{dt} = -\frac{1}{t^2}$	At $t = 1$, $\frac{dy}{dx} = -\frac{2}{3}$
$t = 1$ gives the point $(3, 3)$	$x = \frac{1}{t} + 4t$	$y = 2 - t$	$\frac{dx}{dt} = 4 + 2t$	$\frac{dy}{dt} = -1$	At $t = 1$, $\frac{dy}{dx} = 6$
$t = 1$ gives the point $(4, 1)$	$x = 3t + 1$	$y = \frac{1}{t} + 1$	$\frac{dx}{dt} = 4 - \frac{1}{t^2}$	$\frac{dy}{dt} = -1$	At $t = 1$, $\frac{dy}{dx} = -\frac{1}{3}$

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