Six new gems that teachers of C3 and C4 should know

MEI Conference 2010

The six gems

- Proof: My favourite proof
- Scalar Product and Vector: Cool applications, vector spaces with dimension greater than three
- Differential Equations: Introducing e and giving DEs meaning
- Algebra: The Binomial Expansion
- Functions: Why they save time
- Trigonometry: A different approach? Some great results

Each of these sections contains a gem!

Proof

My favourite proof

Theorem:
There are arbitrarily long sequences of consecutive numbers that are non-prime.

Proof:
For any natural number n larger than 1, the sequence

$$n! + 2,\ n! + 3,\ \ldots,\ n! + n$$

is a sequence of $n - 1$ consecutive composite integers.

Conjectures about primes

- There infinitely many twin primes, pairs of primes with difference 2.
- Or stronger - Polignac's conjecture:for every positive integer n, there are infinitely many pairs of consecutive primes which differ by $2n$.
- There are infinitely many primes of the form $n^2 + 1$
- Brocard's conjecture says that there are always at least four primes between the squares of consecutive primes greater than 2.
- Legendre's conjecture states that there is a prime number between n^2 and $(n + 1)^2$ for every positive integer n.
- Goldbach's conjecture asserts that every even integer greater than 2 can be written as a sum of two primes

The scalar product

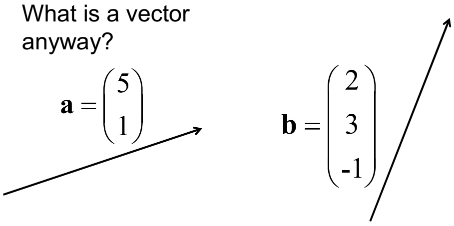
Lighting a scene - world without lighting effects

World with lighting

Use of vectors in lighting

What is a vector anyway?

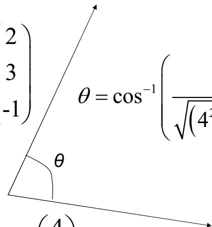
$$\mathbf{a} = \begin{pmatrix} 5 \\ 1 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$$



…we can calculate the angle between two vectors….

$$\mathbf{b} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$$

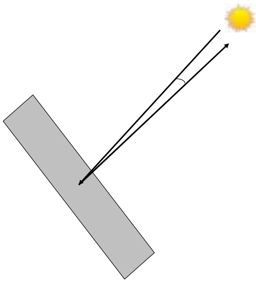
$$\theta = \cos^{-1}\left(\frac{(4\times 2)+(1\times 3)+(2\times -1)}{\sqrt{\left(4^2+1^2+2^2\right)\times\left(2^2+3^2+(-1)^2\right)}}\right)$$



This uses a technique called the scalar product

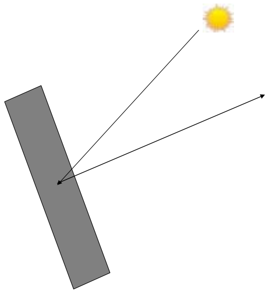
$$\mathbf{a} = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$$

Use of vectors in lighting

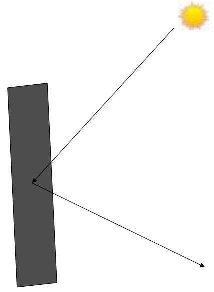


Angle between normal vector and light source determines how light the surface should appear

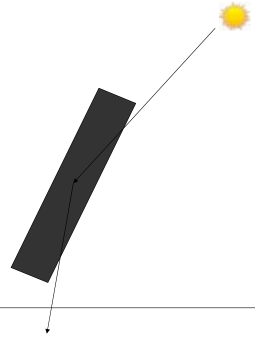
Use of vectors in lighting



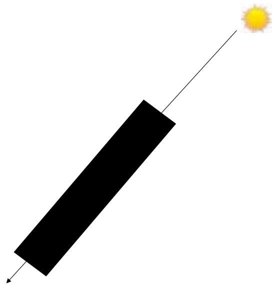
Use of vectors in lighting



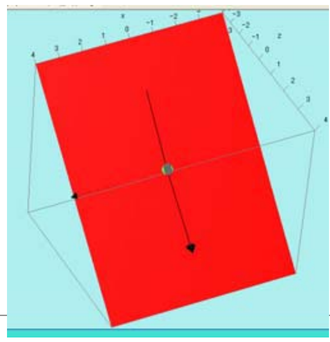
Use of vectors in lighting



Use of vectors in lighting

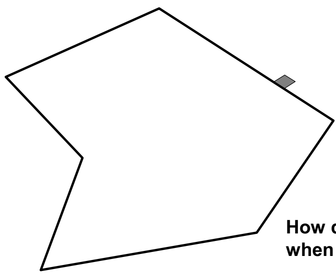


Lighting a Flat Surface



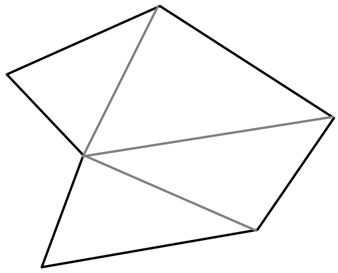
Collision Detection

Collision Detection

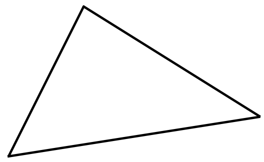


How do we work out when this happens?

Collision Detection



Collision Detection



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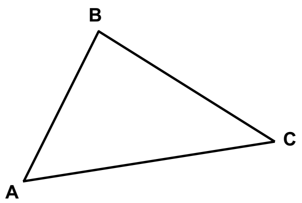

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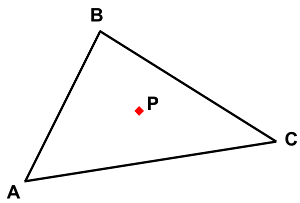

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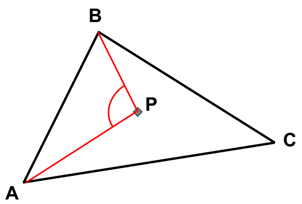

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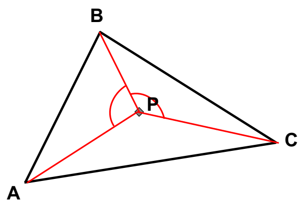

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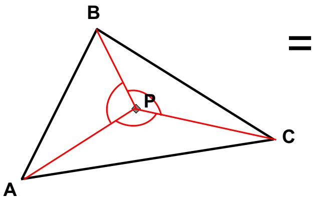

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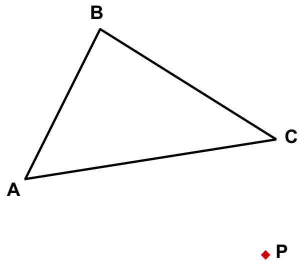
Collision Detection



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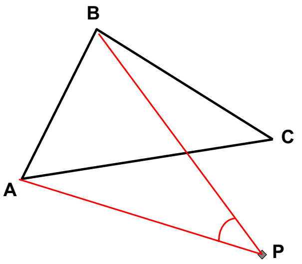

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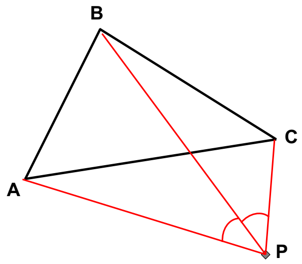

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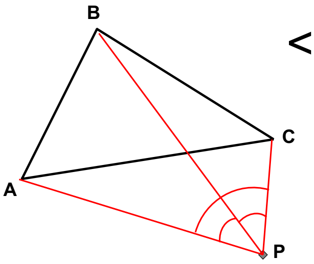

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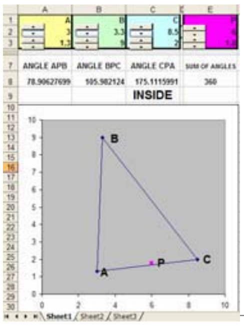
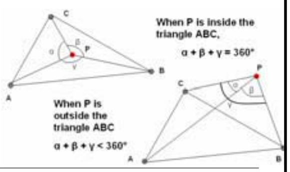
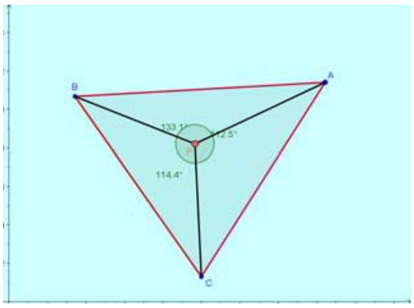
Demonstration – 2D





Geogebra version



Demonstration – 3D

Social networking websites

Compatibility

- Sport - S
- Music - M
- Theatre - T
- Cinema - C

Every person has a vector associated with themselves representing their taste from their application form

Joe and Jenny

- For example:
  Joe's vector is 5S + 2M + 1T – 3C $\begin{pmatrix} 5 \\ 2 \\ 1 \\ -3 \end{pmatrix}$

- Jenny's vector is 2S + 1M + 0T – C $\begin{pmatrix} 2 \\ 1 \\ 0 \\ -1 \end{pmatrix}$

Joe 4 Jenny?

Joe

$$\left|\begin{pmatrix} 5 \\ 2 \\ 1 \\ -3 \end{pmatrix}\right| = \sqrt{5^2 + 2^2 + 1^2 + (-3)^2} = \sqrt{39}$$

Jenny

$$\left|\begin{pmatrix} 2 \\ 1 \\ 0 \\ -1 \end{pmatrix}\right| = \sqrt{2^2 + 1^2 + 0^2 + (-1)^2} = \sqrt{6}$$

Joe■Jenny $\begin{pmatrix} 5 \\ 2 \\ 1 \\ -3 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 1 \\ 0 \\ -1 \end{pmatrix} = (5\times 2) + (2\times 1) + (1\times 0) + (-3\times -1) = 13$

A match!

$$\frac{13}{\sqrt{39}\sqrt{6}} = 0.8498...$$

Joe loves Jenny 84.98%

- Recommendations to customers according to taste on e-commerce sites.

3. Differential Equations

Why do we need differential equations?

- Differential Equations are at the core of calculus.
- Traditionally the key to understanding the physical sciences
- Also one of the most essential practical tools engineers, economists and others have for dealing with rates of change.
- E.g. is C is the cost of producing X widgets then dC/dX is the marginal cost of producing the Xth widget

Differential Equations

- The idea in solving an algebraic equations is to determine from conditions on a number what that number might be.
- The idea in solving a differential equation is to determine from conditions on the derivative (and higher derivatives) of a changing quantity what that quantity might be at any given time.

Differential Equations

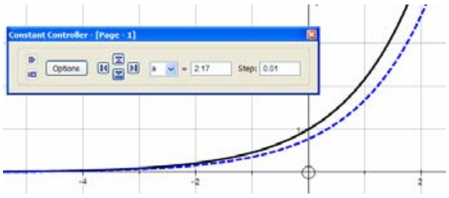
- Newton's laws of motion
- Laplace's heat and wave equations
- Maxwell's electromagnetic theory
- Navier Stokes Equations for fluid dynamics
- Volterra's predator prey systems

The emphasis now is more on numerical approximation and computing and less on traditional methods involving limits and infinite processes.

Useful classroom demonstration

- Students don't seem to necessarily understand what they are doing when they solve a differential equation.

Ways to introduce ‘e’



Ways to introduce ‘e’

- Invest £1 at a rate of 100% interest per annum.
- If the interest is given once at the end of the year then you get £2 back at the end of the year.
- If you get 50% interest after sixth months and then another 50% at the end of the year you get back 1x1.5x1.5 = £2.25

Ways to introduce ‘e’

- If you get 25% every three months you get back 1 x 1.25 x 1.25 x 1.25 x 1.25 = £2.44
- With 12.5% every 1.5 months you get 1.125^8 = £2.57
- Does this sequence keep getting bigger and bigger? Is it bounded? If not, prove it. If it is, does it tend to a limit?

‘e’

- This shows us e’s key role in banking and compound interest calculations.
- ‘e’ makes lots of implausible appearances in problems.
- e is ubiquitous in mathematical formula, theorems and their proofs. It is intimately involved with trigonometric functions, geometrical figures, differential equations, infinite series.

4. Algebra and the binomial theorem

Pascal’s triangle and the binomial theorem

- The additive property of Pascal’s triangle becomes clear when multiplication is set out in a formal way

$$
\begin{array}{r}
a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 \\
a + b \\
\hline
a^5 + 4a^4b + 6a^3b^2 + 4a^2b^3 + ab^4 \\
a^4b + 4a^3b^2 + 6a^2b^3 + 4ab^4 + b^5 \\
\hline
a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5
\end{array}
$$

Extending Pascal's Triangle

- Newton had the remarkable insight that the binomial theorem is not only true for positive integers but also for negative and fractional values of n.
- Lines corresponding to negative integers can added to Pascal's triangle

Extending Pascal's Triangle

1	-2	3	-4	5	-6
1	-1	1	-1	1	-1
1	0	0	0	0	0
1	1	0	0	0	0
1	2	1	0	0	0
1	3	3	1	0	0
1	4	6	4	1	0
1	5	10	10	5	1

$$\frac{1}{1+x} = (1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 -$$

$$\frac{1}{(1+x)^2} = (1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + 5x^4 -$$

5. Functions

Functions

- One important aspect of functions that is often overlooked is that the enable us to save time.
- For example the chain rule, the product rule and the quotient rule mean that we can differentiate a huge class of functions from just knowing how to differentiate simple ones.
- A great example of this is Lighbot http://armorgames.com/play/2205/light-bot

6. Trigonometry

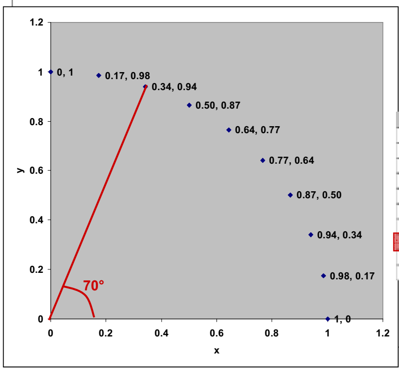
Approaches to trigonometry

- Usually trigonometry is introduced as being concerned with the ratio of side lengths in right angled triangles
- Ratio is a difficult concept algebraically and an approach based on similar triangles and enlargement might be better…
- Sometimes over-use of algebra is not the best approach

Sin and Cos

- Introduce them by getting students to see the output for various values of θ.

θ	cosθ	sinθ
0	1	0
10	0.98	0.17
20	0.94	0.34
30	0.87	0.50
40	0.77	0.64
50	0.64	0.77
60	0.50	0.87
70	0.34	0.94
80	0.17	0.98
90	0	1



θ	cosθ	sinθ
0	1	0
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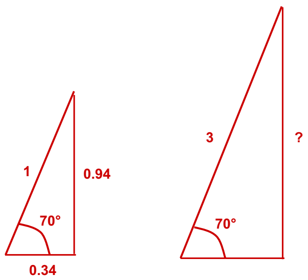
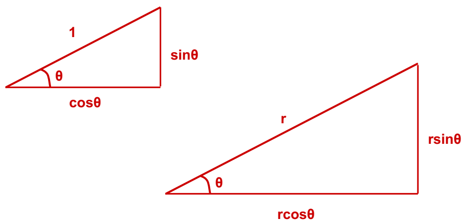
30	0.87	0.50
40	0.77	0.64
50	0.64	0.77
60	0.50	0.87
70	0.34	0.94
80	0.17	0.98
90	0	1

Doing problems using an enlargement technique



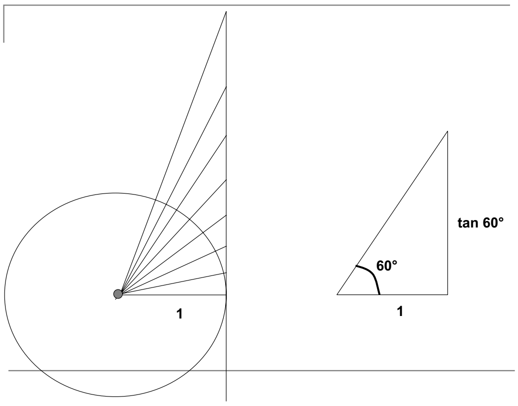
This then follows very naturally….



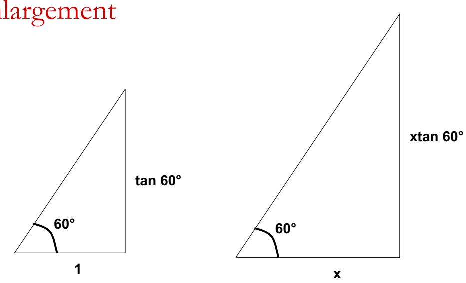
Contrast the approaches

Tan and Gradient

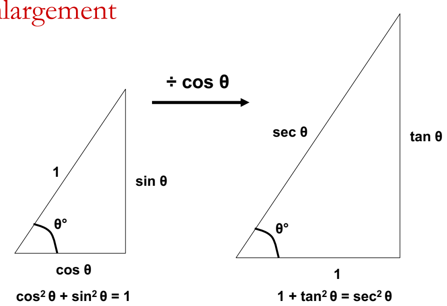
- Tan often gets overlooked in A-level mathematics and is frequently dealt with as sin/cos.
- Giving tan attention in its own right and looking at the connection to gradient can pay dividends.



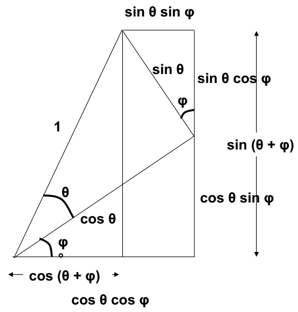
Defining tan as a length using enlargement



Defining tan as a length using enlargement



Geometric proofs



Trigonometry

- Draw the graph of y = $\sin^2 x$
- Is this a translation of sinx?