Markov Chains: a great FP3 option and an enrichment topic accessible to all

MEI Conference 2011

Matrices and Matrix Multiplication

- Introduction
- Use excel to get the product of two 2x2 matrices and then two 3x3 matrices.

Why is Markov Chains a good option in FP3

- Beautiful application of matrices that complements FP1 and FP2 well
- Modern applications
- Students who are well-prepared for this question will invariably be successful on it in the examination

What is a Markov Chain

- It’s a system that has a number of states and it moves from state to state with certain probabilities.

Why does Markov Chains make for great enrichment?

- Straightforward...
- New topics
- Spreadsheets
- Probability in the long term
- Surprising results
- Lots of experimentation possible
- Computer simulation
- Definitions and Proof
- Modern applications that are meaningful to young students

Typical First example

- An island

If it’s sunny today the probability that it will be sunny tomorrow is 0.7 and the probability that it rains is 0.3.

If it rains today the probability that it will rain tomorrow is 0.8 and the probability that it will be sunny is 0.2
Transition Matrix

\[
\begin{pmatrix}
0.7 & 0.2 \\
0.3 & 0.8
\end{pmatrix}
\]

Today

<table>
<thead>
<tr>
<th></th>
<th>Sun</th>
<th>Rain</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sun</td>
<td>0.7</td>
<td>0.3</td>
</tr>
<tr>
<td>Rain</td>
<td>0.2</td>
<td>0.8</td>
</tr>
</tbody>
</table>

Tomorrow

Sunny today

\[
\begin{pmatrix}
0.7 \\
0.3
\end{pmatrix}
\]

The next day

\[
\begin{pmatrix}
0.7 \\
0.3
\end{pmatrix} \times \begin{pmatrix}
0.7 & 0.2 \\
0.3 & 0.8
\end{pmatrix} = \begin{pmatrix}
0.7 \times 0.7 + 0.3 \times 0.2 \\
0.7 \times 0.3 + 0.3 \times 0.8
\end{pmatrix}
\]

Rainy today

\[
\begin{pmatrix}
0.2 \\
0.8
\end{pmatrix}
\]

The next day

\[
\begin{pmatrix}
0.2 & 0.8 \\
0.2 & 0.8
\end{pmatrix}
\]

\[
\begin{pmatrix}
0.2 \times 0.7 + 0.8 \times 0.2 \\
0.2 \times 0.3 + 0.8 \times 0.8
\end{pmatrix}
\]

Matrix Multiplication

\[
\begin{pmatrix}
0.7 & 0.2 \\
0.3 & 0.8
\end{pmatrix}^2
\]

\[
\begin{pmatrix}
0.7 \\
0.3
\end{pmatrix}
\]

Thursday

P\{Sunny on Sat\} = \begin{pmatrix}
0.7 & 0.2 \\
0.3 & 0.8
\end{pmatrix} \times \begin{pmatrix}
0.7 \\
0.3
\end{pmatrix} = 0.7 \times 0.7 + 0.3 \times 0.2

P\{Rain on Sat\} = \begin{pmatrix}
0.7 & 0.2 \\
0.3 & 0.8
\end{pmatrix} \times \begin{pmatrix}
0.7 \\
0.3
\end{pmatrix} = 0.7 \times 0.3 + 0.3 \times 0.8
Matrix Multiplication

Thursday

Sunday

$$\begin{pmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{pmatrix}^3$$

Matrix Multiplication

Thursday

Monday

$$\begin{pmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{pmatrix}^4$$

How could we use Excel to calculate high powers of a matrix quickly?

Equilibrium Probabilities

$$A^{10} = \begin{pmatrix} 0.4005859375 & 0.399609375 \\ 0.5994140625 & 0.600390625 \end{pmatrix}$$

$$A^{100} = \begin{pmatrix} 0.4 & 0.4 \\ 0.6 & 0.6 \end{pmatrix}$$

$$A^{1000} = \begin{pmatrix} 0.4 & 0.4 \\ 0.6 & 0.6 \end{pmatrix}$$

$$A^{10000} = \begin{pmatrix} 0.4 & 0.4 \\ 0.6 & 0.6 \end{pmatrix}$$

What are these matrices?

$$A^2 = \begin{pmatrix} 0.55 & 0.3 \\ 0.45 & 0.7 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 0.475 & 0.35 \\ 0.525 & 0.65 \end{pmatrix}$$

$$A^4 = \begin{pmatrix} 0.4375 & 0.375 \\ 0.5625 & 0.625 \end{pmatrix}$$

$$A^5 = \begin{pmatrix} 0.41875 & 0.3875 \\ 0.58125 & 0.6125 \end{pmatrix}$$

Ways to find equilibrium probabilities

- High powers of the matrix
- Fixed Point iteration
- Simulation
- Algebra
Modelling a situation with a Markov Chain

- Modelling is a skill that can be very weak in students.
- Markov chains is a good way to introduce modelling because the mathematics isn’t too difficult.

Example - Modelling

- We have two urns that, between them, contain four balls.
- At each step, one of the four balls is chosen at random and moved from the urn that it is in into the other urn.
- We choose, as states, the number of balls in the first urn.
- What is the transition matrix?

Tennis

- Consider the game of tennis at Wimbledon in the final set at 6-6.
- When A is serving she has a 0.99 chance of winning the game. When B is serving she has a 0.99 chance of winning the game.
- Set this up as a Markov chain with.

Drunkard’s Walk

\[
\begin{pmatrix}
1 & 0.5 & 0 & 0 \\
0 & 0 & 0.5 & 0 \\
0 & 0.5 & 0 & 0.5 \\
0 & 0 & 0.5 & 0 \\
0 & 0 & 0 & 0.5 & 1
\end{pmatrix}
\]

Simulating a Markov Chain
Use of dice, connections with Simulation chapter of D1

- Students should use dice/coins/random number generators to determine path through the system an instance.
- You should decide how many iterations an instance should live for until its final position is recorded.
- You’ll need to think about initial conditions.

Other things to investigate

- Run length
- Absorbing States
- Reflective Barriers
- Regular Chains
- Ergodic Chains
- Periodic Chains

Expected run length

The expected run length for rainy days would be

\[(0 \times 0.2) + (1 \times 0.8 \times 0.2) + (2 \times 0.8^2 \times 0.2) + \ldots\]

\[= 0.8 \times 0.2 \left(1 + 2 \times 0.8 + 3 \times 0.8^2 + \ldots\right)\]

\[= (0.8 \times 0.2) \frac{1}{1 - 0.8} \]

\[= \frac{0.16}{0.2} \]

Run length

- Give a Markov Chain what is the expected number of times it will stay in a particular state?

Expected run length

- In general the expected run length is

\[E(X) = \frac{1}{1 - \alpha} \times \frac{1}{1 - \alpha} \times \frac{\alpha}{1 - \alpha}\]

- There are lots of questions you might ask students about this (about the effects of changing \(\alpha\))
Absorbing State

- If a Markov chain has an absorbing state then eventually the system will go into one of the absorbing states.

Absorbing State Example - Drunkard’s Walk

- Absorbing States – a state you can’t leave
- Transient States – all other states

Reflecting Barrier

- When the probability of changing from state i to state j is 1, state i is said to be a reflecting barrier.
- Questions for students: what would the transition matrix look like when there is a reflecting barrier?

Periodic chain

- A periodic chain is one for which the is a value of k such that $P^k = P$, where P is the transition matrix of the chain.
- The period of such a chain is then said to be $k - 1$ where k is the smallest such value.

More terminology

- A Markov chain is called an ergodic chain if it is possible to go from every state to every state
- A Markov chain is called regular if some power of the transition matrix has only strictly positive entries

Questions

- Is an ergodic chain necessarily regular?
- Is an regular chain necessarily ergodic?
Answers

- A regular chain must be ergodic. The fact that a power of the transition matrix has entirely positive entries shows that it must be possible to go from any state to any other state.
- Ergodic does not imply regular. Can you think of a counter example?

The Google Empire Today

- Google Docs
- Google Analytics
- Google Earth
- Google Maps
- Google Desktop Search
- Gmail
- Google Image Search

Founders of Google

Larry Page
Sergey Brin

Students at Stanford University, California whose search engine at the time was nicknamed 'Backrub'.

We asked 100 people - why is Google the world's most popular search engine?

Google: Number of searches

December 2009:

- **Google**: 88 billion per month
- **Twitter**: 19 billion per month
- **Yahoo**: 9.4 billion per month
- **Bing**: 4.1 billion per month
1. The way it sorts results
2.
3. 03

1. The way it sorts results
2. Speed
3. 03

1. Speed
2. Speed
3. Because it's cool! 03

1. Speed
2.
3. Because it's cool! 03

1. The way it sorts results
2.
3. Because it's cool! 03
The mathematical secrets of Google

- You're about to learn the mathematical secret that was the basis for Page and Brin's brilliant idea and how this links to Markov Chains.
- This is extremely valuable information.
An improved PageRank

What are the equilibrium probabilities for this network?

Pages 6 and 7 absorb all of the importance that should have been divided up amongst the other pages.

Solution

- One approach is just to remove the offending pages. This is not practical parts of the web that act like this might consist of thousands of pages.
- A surfer stuck in a loop would grow bored and decide to visit another part of the web at random.
- Inventors of page rank suggested adding to $P$ a matrix $Q$ that represents the taste of the impartial web surfer.
Solution

\[ P' = \beta P + (1 - \beta)Q \]
where \( 0 \leq \beta \leq 1 \)

Using \( Q \) where every element is \( 1/7 \) we get....
FP3 14th June 2007

Question 5 (Part Two)

(i) Find the probability that the 12th letter is the same as the 6th letter.

(ii) By investigating the behaviour of $P^n$ when $n$ is large, find the probability that the $1 + k$th letter is a when

(iii) $n$ is a large even number.

(iv) $n$ is a large odd number.

The program is now changed. The initial probabilities and the transition probabilities are the same as before, except for the following. After $D$, the next letter is $A$, $B$, or $D$, with probabilities 0.3, 0.4, and 0.3 respectively.

(v) Write down the new transition matrix $Q$.

(vi) Verify that $Q^n$ approaches a limit as $n$ becomes large, and hence write down the equilibrium probabilities for $A$, $B$, and $D$.

(vii) When $n$ is large, find the probability that the $(1 + k)$th, $1 + (2k)$th, and $1 + (3k)$th letters are

100.